

# Deformation of Hypersurfaces Preserving the Möbius Metric and a Reduction Theorem

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## Abstract

A hypersurface without umbilics in the  $(n + 1)$ -dimensional Euclidean space  $f : M^n \rightarrow R^{n+1}$  is known to be determined by the Möbius metric  $g$  and the Möbius second fundamental form  $B$  up to a Möbius transformation when  $n \geq 3$ . In this paper we consider Möbius rigidity for hypersurfaces and deformations of a hypersurface preserving the Möbius metric in the high dimensional case  $n \geq 4$ . When the highest multiplicity of principal curvatures is less than  $n - 2$ , the hypersurface is Möbius rigid. Deformable hypersurfaces and the possible deformations are also classified completely. In addition, we establish a Reduction Theorem characterizing the classical construction of cylinders, cones, and rotational hypersurfaces, which helps to find all the non-trivial deformable examples in our classification with wider application in the future.

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## 1 Introduction

In submanifold theory a fundamental problem is to investigate which data are sufficient to determine a submanifold  $M$  up to the action of a certain transformation group

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$G$  on the ambient space. The *deformable case* means that there exists non-congruent (depending on  $G$ ) immersions with the same given invariants at corresponding points, and such different immersions are called *deformations* to each other. In contrast, the *rigid case* indicates that such deformations do not exist (or just be congruent by the action of  $G$ ). In this paper we consider deformations of hypersurfaces  $M^n$  preserving the so-called Möbius metric in the framework of Möbius geometry ( $G$  is the Möbius transformation group acting on  $R^{n+1} \cup \{\infty\}$ ).

As a background let us review some classical results. It is known that a generic immersed surface in Euclidean three-space  $u : M^2 \rightarrow R^3$  is determined, up to a rigid motion of  $R^3$ , by its induced metric  $I$  and mean curvature function  $H$ . All exceptional immersions are called *Bonnet surfaces*, which were classified by Bonnet[2], Cartan[7] and Chern[8] into three distinct classes:

- 1) CMC (constant mean curvature) surfaces with a 1-parameter deformations preserving  $I$  and  $H$  (known as *the associated family*);
- 2) Not CMC and admits a continuous 1-parameter deformations;
- 3) Surfaces that admit exactly one such deformation.

In either case, two Bonnet surfaces forming deformation to each other is called a *Bonnet pair*. These notions are directly generalized to other space forms  $S^3$  and  $H^3$ . See [1, 16, 17, 22] for recent works on this topic.

For a hypersurface  $f : M^n \rightarrow R^{n+1}$  ( $n \geq 3$ ), the well-known Beez-Killing rigidity theorem says that  $f$  is isometrically rigid if the rank of its second fundamental form (i.e. the number of non-zero principal curvatures) is greater than or equal to 3 everywhere. Compared to surface case this is a stronger rigidity result, mainly due to the Gauss equations which forms an over-determined system when there are many non-zero principal curvatures.

On the other hand, all isometrically deformable hypersurfaces have rank 2 or less. They are locally classified by Sbrana [21] and Cartan [3]. According to their results there are four classes of them. The first two classes (surface-like and ruled) are highly deformable. The third class admits precisely a continuous 1-parameter family of deformations, and the fourth class has a unique deformation.

In Möbius geometry, let  $f, \bar{f} : M^n \rightarrow R^{n+1}$  be two hypersurfaces in the  $(n+1)$ -dimensional Euclidean space  $R^{n+1}$ . We say  $f$  is Möbius equivalent to  $\bar{f}$  (or  $f$  is Möbius congruent to  $\bar{f}$ ) if there exists a Möbius transformation  $\Psi$  such that  $f = \Psi \circ \bar{f}$ . It is natural to consider deformations preserving certain conformal invariants. In [4]

Cartan considered the problem of *conformal deformation*, i.e. deformation of any given hypersurface preserving the conformal class of the induced metric. Cartan has given the following conformal rigidity result:

**Theorem 1.1.** [4] *A hypersurface  $f : M^n \rightarrow R^{n+1}$  ( $n \geq 5$ ) is conformally rigid if each principal curvature has multiplicity less than  $n - 2$  everywhere.*

In [11] do Carmo and Dajczer generalized Cartan's rigidity theorem to submanifolds of dimension  $n \geq 5$ . Note that the multiplicity of a principal curvature is Möbius invariant. When the highest multiplicity is  $n$  or  $n - 1$ , it is the conformally flat case well-known to be highly deformable. When  $n \geq 5$  and the highest multiplicity is  $n - 2$ , Cartan [4] gave a quite similar classification of conformally deformable hypersurfaces into four cases:

- I) Surface-like hypersurfaces (which are cylinders, cones or revolution hypersurfaces over surfaces in 3-dim space forms);
- II) Conformally ruled hypersurfaces;
- III) One of those having a continuous 1-parameter family of deformations;
- IV) One of those that admits a unique deformation.

In [9] and [10] Dajczer et.al. gave a modern account of Sbrana and Cartan's classification. Following Dajczer, we call such conformally deformable hypersurfaces as *Cartan hypersurfaces* of class I, II, III, and IV.

We observe that in the conformal class of a given immersed hypersurface in  $R^{n+1}$  there is a distinguished metric called the *Möbius metric*  $g$ . Together with the *Möbius second fundamental form*  $B$  they form a complete system of invariants in Möbius geometry (see [24] or Theorem 2.2 in this paper). Based on our experience, the deformation preserving the Möbius metric  $g$  seems to be a natural and new topic.

**Definition 1.2.** *A hypersurface  $f : M^n \rightarrow R^{n+1}$  is said to be Möbius rigid if any other immersion  $\bar{f} : M^n \rightarrow R^{n+1}$  sharing the same Möbius metric  $g$  as  $f$ , is Möbius equivalent to  $f$ . An immersion  $\bar{f} : M^n \rightarrow R^{n+1}$  is said to be a Möbius deformation of  $f$  if they induce the same Möbius metric  $g$  at corresponding points and  $\bar{f}(M)$  is not congruent to  $f(M)$  up to any Möbius transformation.*

We obtain the following Möbius Rigidity Theorem.

**Theorem 1.3.** *Let  $f : M^n \rightarrow R^{n+1}$  ( $n \geq 4$ ) be a hypersurface in the  $(n + 1)$ -dimensional Euclidean space. If any principal curvature of  $f$  has multiplicity less than  $n - 2$  everywhere, then  $f$  is Möbius rigid.*

**Remark 1.4.** Compared with Cartan's notion before, a conformally rigid hypersurface  $f : M^n \rightarrow R^{n+1}$  is Möbius rigid, but the converse may not be true. On the other hand, if  $f$  is Möbius deformable with deformation  $\bar{f}$ , they are also conformal deformations to each other, but the converse may not be true. Thus when  $n \geq 5$  our rigidity theorem is a corollary of Cartan's conformally rigidity result.

On the other hand, Cartan treated the special dimensions  $n = 4, 3$  in [5, 6]. In particular, in [5] Cartan has shown that, for  $n = 4$ , there exist hypersurfaces  $f, \bar{f} : M^4 \rightarrow R^5$  that have four distinct principal curvatures at each point  $p \in M^4$  and are conformal deformations to each other. In contrast, our Möbius rigidity result as above still holds true for dimension  $n = 4$ . Because of this interesting difference and for the purpose of self-containedness we give a proof to Theorem 1.3 in Section 7.

The main result of this paper is the following classification theorem of all Möbius deformable hypersurfaces.

**Theorem 1.5.** Let  $f : M^n \rightarrow R^{n+1}$  ( $n \geq 4$ ) be a hypersurface in the  $(n + 1)$ -dimensional Euclidean space without umbilics. Suppose  $f$  is Möbius deformable.

1) When one principal curvature of  $f$  has multiplicity  $n - 1$  everywhere, this deformable  $f$  must have constant Möbius sectional curvature. They are either cones, cylinders or rotational hypersurfaces over the so-called curvature-spirals in 2-dimensional space-forms. (See [13] for the classification or Section 4 for an independent proof.)

2) When one principal curvature of  $f$  has multiplicity  $n - 2$  everywhere, locally  $f$  is Möbius equivalent to either of the three classes below:

(a)  $f(M^n) \subset L^2 \times R^{n-2}$ , where  $L^2$  is a Bonnet surface in  $R^3$ ;

(b)  $f(M^n) \subset CL^2 \times R^{n-3}$ , where  $CL^2 \subset R^4$  is a cone over  $L^2 \subset S^3$ , and  $L^2$  is a Bonnet surface in  $S^3$ ;

(c)  $f(M^n)$  is a rotational hypersurface over  $L^2 \subset R_+^3$ , where  $L^2$  is a Bonnet surface in the hyperbolic half space model  $R_+^3$ .

Moreover, the Möbius deformation to any of them belongs to the same class and comes from the deformation of the corresponding Bonnet surface  $L^2$ .

**Remark 1.6.** According to our classification result, among Cartan hypersurfaces [10], only the first class (surface-like hypersurfaces) may share the same Möbius metric with their conformal deformations. The other three classes of conformally deformable hypersurfaces are Möbius rigid in our sense.

**Remark 1.7.** In the definition above, it is noteworthy that the non-congruence between  $\bar{f}(M), f(M)$  (the images) is stronger than the non-congruence between  $\bar{f}, f$  (the

mappings), because the same hypersurface  $f(M) \subset R^n$  might be given different parameterizations  $f$  and  $\bar{f}$  which are NOT Möbius equivalent. In other words there might exist an (isometrical) diffeomorphism  $\phi : M^n \rightarrow M^n$  and a Möbius transformation  $\Psi : R^{n+1} \cup \{\infty\} \rightarrow R^{n+1} \cup \{\infty\}$  such that the following diagram commutes:

$$\begin{array}{ccc} M^n & \xrightarrow{\psi} & M^n \\ f \downarrow & & \downarrow \bar{f} \\ R^{n+1} \cup \{\infty\} & \xrightarrow{\Psi} & R^{n+1} \cup \{\infty\} \end{array}$$

A typical example is the Möbius isoparametric hypersurface (see [19, 15] or Section 9 for the definition) with three distinct constant Möbius principal curvatures

$$\sqrt{\frac{n-1}{2n}}, -\sqrt{\frac{n-1}{2n}}, 0, \dots, 0.$$

It is part of the cone over the Cartan minimal isoparametric hypersurface  $y : N^3 \rightarrow S^4(1) \hookrightarrow R^5 \subset R^{n+1}$  with three distinct principal curvatures. This  $N^3$  is a tube of a specific constant radius over the Veronese embedding  $RP^2 \hookrightarrow S^4$ . It is well-known that its induced metric has a 4-dimensional isometry group whose elements do NOT preserve the principal distributions in general. Any such isometry  $\phi$  extends to an isometry of the cone (with respect to its Möbius metric  $g$ ) which is surely NOT a Möbius transformation of the ambient space. Any possible deformation  $\bar{f}$  to the cone  $f$  preserving Möbius metric  $g$  arises in this way, hence is excluded from our notion (as well as the classification list) of Möbius deformable hypersurfaces. See the discussion of this example in Section 9.

**Remark 1.8.** For a hypersurface  $f : M \rightarrow R^{n+1}$  of constant Möbius curvature  $c$ , generally we can map any neighborhood of a given point  $p \in M$  to a neighborhood of another point  $q \in M$  by an isometry (of  $(M, g)$ ) which is not induced from a Möbius transformation of the ambient space. This is because any such hypersurface is conformally flat with a specific principal direction which is not preserved by a generic isometry of  $(M, g)$ . So they provide the first class of deformable hypersurfaces.

Circular cylinder and spiral cylinder (constructed from a circle or a logarithmic spiral, respectively) belong to this class, yet they are different. Each of them is homogeneous, namely invariant under a subgroup of the Möbius group (of dimension at least  $n$ ) which acts transitively on  $M^n$ . On the other hand each of them have a bigger isometry group (with respect to  $(M^n, g)$ ) which generally are not induced from Möbius transformations. So they resemble Cartan's example in the previous remark. Yet these two hypersurfaces still have non-trivial deformations. See final remarks in Section 4.

**Remark 1.9.** *Some comments on low dimensional case  $n = 3$  or  $2$ . We do not have any Möbius rigidity result because our algebraic theorem 6.1 fails in this case (see Remark 6.3). But the construction of Möbius deformable hypersurfaces in Section 3 is still valid for  $n = 3$ .*

*When  $n = 2$ , generally a surface with a given Möbius metric is highly deformable. So we would consider deformation problems under stronger restrictions. We just mention that any Willmore surface admits a one-parameter associated family of Willmore surfaces endowed with the same Möbius metric. For more on related topics see [12].*

**Remark 1.10.** *It is very interesting that the non-trivial deformable examples all arise from the classical construction of cylinders, cones or rotational hypersurfaces over a given hypersurface in a low-dimensional Euclidean subspace, sphere or hyperbolic half-space, respectively. Such constructions appeared many times in various contexts and problems in Möbius geometry and Lie sphere geometry. We find that such examples have a nice characterization (Theorem 5.1) in terms of its Möbius invariants introduced by the third author in [24]. We believe that this Reduction Theorem is a valuable tool in simplifying discussions of many similar problems.*

We organize the paper as follows. In Section 2, we introduce Möbius invariants and the Möbius congruence theorem for hypersurfaces in  $R^{n+1}$  ( $n \geq 3$ ). Examples of Möbius deformable hypersurfaces are given in Section 3 and 4 (in particular, Section 4 gives a new proof to the classification theorem of hypersurfaces with constant Möbius sectional curvature). These examples are characterized by our Reduction Theorems 5.1 (used in Section 9) and 5.3 (used in Section 4) proved in Section 5.

After these preparations, as a purely algebraic consequence of the Gauss equation we show in Section 6 that the (Möbius) second fundamental forms of  $f$  and its deformation  $\bar{f}$  could be diagonalized almost simultaneously. Then we investigate our problem case by case. When the highest multiplicity is less than  $n - 2$  we establish the rigidity result (Theorem 1.3) in Section 7. Section 8 treats the conformally flat case (i.e. the highest multiplicity is equal to  $n - 1$ ) where we show such deformable examples must have constant Möbius curvature, which have been classified in Section 4. In Section 9 all deformable hypersurfaces with one principal curvature of multiplicity  $n - 2$  are proved to be reducible to cylinders, cones or rotational hypersurfaces using the Reduction Theorem in Section 5. This finishes the proof to the Main Theorem 1.5.

## 2 Möbius invariants for hypersurfaces in $R^{n+1}$

In this section we briefly review the theory of hypersurfaces in Möbius geometry. For details we refer to [24], [18].

Let  $R_1^{n+3}$  be the Lorentz space, i.e.,  $R^{n+3}$  with inner product  $\langle \cdot, \cdot \rangle$  defined by

$$\langle x, y \rangle = -x_0y_0 + x_1y_1 + \cdots + x_{n+2}y_{n+2},$$

for  $x = (x_0, x_1, \dots, x_{n+2}), y = (y_0, y_1, \dots, y_{n+2}) \in R^{n+3}$ .

Let  $f : M^n \rightarrow R^{n+1}$  be a hypersurface without umbilics and assume that  $\{e_i\}$  is an orthonormal basis with respect to the induced metric  $I = df \cdot df$  with  $\{\theta_i\}$  the dual basis. Let  $II = \sum_{ij} h_{ij} \theta_i \theta_j$  and  $H = \sum_i \frac{h_{ii}}{n}$  be the second fundamental form and the mean curvature of  $f$ , respectively. We define the Möbius position vector  $Y : M^n \rightarrow R_1^{n+3}$  of  $f$  by

$$Y = \rho \left( \frac{1 + |f|^2}{2}, \frac{1 - |f|^2}{2}, f \right), \quad \rho^2 = \frac{n}{n-1} (|II|^2 - nH^2).$$

**Theorem 2.1.** [24] *Two hypersurfaces  $f, \bar{f} : M^n \rightarrow R^{n+1}$  are Möbius equivalent if and only if there exists  $T$  in the Lorentz group  $O(n+2, 1)$  in  $R_1^{n+3}$  such that  $\bar{Y} = YT$ .*

It follows immediately from Theorem 2.1 that

$$g = \langle dY, dY \rangle = \rho^2 df \cdot df$$

is a Möbius invariant, called the Möbius metric of  $f$ .

Let  $\Delta$  be the Laplacian with respect to  $g$ . Define

$$N = -\frac{1}{n} \Delta Y - \frac{1}{2n^2} \langle \Delta Y, \Delta Y \rangle Y,$$

which satisfies

$$\langle Y, Y \rangle = 0 = \langle N, N \rangle, \quad \langle N, Y \rangle = 1.$$

Let  $\{E_1, \dots, E_n\}$  be a local orthonormal basis for  $(M^n, g)$  with dual basis  $\{\omega_1, \dots, \omega_n\}$ . Write  $Y_i = E_i(Y)$ . Then we have

$$\langle Y_i, Y \rangle = \langle Y_i, N \rangle = 0, \quad \langle Y_i, Y_j \rangle = \delta_{ij}, \quad 1 \leq i, j \leq n.$$

Let  $\xi$  be the mean curvature sphere of  $f$  written as

$$\xi = \left( \frac{1 + |f|^2}{2} H + f \cdot e_{n+1}, \frac{1 - |f|^2}{2} H - f \cdot e_{n+1}, Hf + e_{n+1} \right),$$

where  $e_{n+1}$  is the unit normal vector field of  $f$  in  $R^{n+1}$ .

Then  $\{Y, N, Y_1, \dots, Y_n, \xi\}$  forms a moving frame in  $R_1^{n+3}$  along  $M^n$ . We will use the following range of indices in this section:  $1 \leq i, j, k \leq n$ . We can write the structure equations as following:

$$\begin{aligned} dY &= \sum_i Y_i \omega_i, \\ dN &= \sum_{ij} A_{ij} \omega_i Y_j + \sum_i C_i \omega_i \xi, \\ dY_i &= - \sum_j A_{ij} \omega_j Y - \omega_i N + \sum_j \omega_{ij} Y_j + \sum_j B_{ij} \omega_j \xi, \\ d\xi &= - \sum_i C_i \omega_i Y - \sum_{ij} \omega_i B_{ij} Y_j, \end{aligned}$$

where  $\omega_{ij}$  is the connection form of the Möbius metric  $g$  and  $\omega_{ij} + \omega_{ji} = 0$ . The tensors

$$\mathbf{A} = \sum_{ij} A_{ij} \omega_i \otimes \omega_j, \quad \mathbf{B} = \sum_{ij} B_{ij} \omega_i \otimes \omega_j, \quad \Phi = \sum_i C_i \omega_i$$

are called the Blaschke tensor, the Möbius second fundamental form and the Möbius form of  $f$ , respectively. The covariant derivative of  $C_i, A_{ij}, B_{ij}$  are defined by

$$\begin{aligned} \sum_j C_{i,j} \omega_j &= dC_i + \sum_j C_j \omega_{ji}, \\ \sum_k A_{ij,k} \omega_k &= dA_{ij} + \sum_k A_{ik} \omega_{kj} + \sum_k A_{kj} \omega_{ki}, \\ \sum_k B_{ij,k} \omega_k &= dB_{ij} + \sum_k B_{ik} \omega_{kj} + \sum_k B_{kj} \omega_{ki}. \end{aligned}$$

The integrability conditions for the structure equations are given by

$$\begin{aligned} (1) \quad & A_{ij,k} - A_{ik,j} = B_{ik} C_j - B_{ij} C_k, \\ (2) \quad & C_{i,j} - C_{j,i} = \sum_k (B_{ik} A_{kj} - B_{jk} A_{ki}), \\ (3) \quad & B_{ij,k} - B_{ik,j} = \delta_{ij} C_k - \delta_{ik} C_j, \\ (4) \quad & R_{ijkl} = B_{ik} B_{jl} - B_{il} B_{jk} + \delta_{ik} A_{jl} + \delta_{jl} A_{ik} - \delta_{il} A_{jk} - \delta_{jk} A_{il}, \\ (5) \quad & R_{ij} := \sum_k R_{ikjk} = - \sum_k B_{ik} B_{kj} + (tr \mathbf{A}) \delta_{ij} + (n-2) A_{ij}, \\ (6) \quad & \sum_i B_{ii} = 0, \sum_{ij} (B_{ij})^2 = \frac{n-1}{n}, tr \mathbf{A} = \sum_i A_{ii} = \frac{1}{2n} (1 + n^2 \kappa), \end{aligned}$$

where  $R_{ijkl}$  denote the curvature tensor of  $g$ ,  $\kappa = \frac{1}{n(n-1)} \sum_{ij} R_{ijij}$  is its normalized Möbius scalar curvature. We know that all coefficients in the structure equations are determined by  $\{g, \mathbf{B}\}$  and we have



**Theorem 2.2.** [24] *Two hypersurfaces  $f : M^n \rightarrow R^{n+1}$  and  $\bar{f} : M^n \rightarrow R^{n+1}$  ( $n \geq 3$ ) are Möbius equivalent if and only if there exists a diffeomorphism  $\varphi : M^n \rightarrow M^n$  which preserves the Möbius metric and the Möbius second fundamental form.*

The second covariant derivative of  $B_{ij}$  are defined by

$$dB_{ij,k} + \sum_m B_{mj,k} \omega_{mi} + \sum_m B_{im,k} \omega_{mj} + \sum_m B_{ij,m} \omega_{mk} = \sum_m B_{ij,km} \omega_m.$$

We have the following Ricci identities

$$B_{ij,kl} - B_{ij,lk} = \sum_m B_{mj} R_{mikl} + \sum_m B_{im} R_{mjkl}.$$

Coefficients of Möbius invariants and Euclidean invariants are related by [18]

$$(7) \quad \begin{aligned} B_{ij} &= \rho^{-1}(h_{ij} - H\delta_{ij}), \\ C_i &= -\rho^{-2}[e_i(H) + \sum_j (h_{ij} - H\delta_{ij})e_j(\log \rho)], \\ A_{ij} &= -\rho^{-2}[Hess_{ij}(\log \rho) - e_i(\log \rho)e_j(\log \rho) - Hh_{ij}] \\ &\quad - \frac{1}{2}\rho^{-2}(|\nabla \log \rho|^2 + H^2)\delta_{ij}, \end{aligned}$$

where  $Hess_{ij}$  and  $\nabla$  are the Hessian matrix and the gradient with respect to  $I = df \cdot df$ . Then

$$A = \rho^2 \sum_{ij} A_{ij} \theta_i \otimes \theta_j, \quad B = \rho^2 \sum_{ij} B_{ij} \theta_i \otimes \theta_j, \quad \Phi = \rho \sum_i C_i \theta_i.$$

We call eigenvalues of  $(B_{ij})$  as *Möbius principal curvatures* of  $f$ . Clearly the number of distinct Möbius principal curvatures is the same as that of its distinct Euclidean principal curvatures.

Let  $k_1, \dots, k_n$  be the principal curvatures of  $f$ , and  $\{\lambda_1, \dots, \lambda_n\}$  the corresponding Möbius principal curvatures, then the curvature sphere of principal curvature  $k_i$  is

$$\xi_i = \lambda_i Y + \xi = \left( \frac{1 + |f|^2}{2} k_i + f \cdot e_{n+1}, \frac{1 - |f|^2}{2} k_i - f \cdot e_{n+1}, k_i f + e_{n+1} \right).$$

Note that  $k_i = 0$  if, and only if,

$$\langle \xi_i, (1, -1, 0, \dots, 0) \rangle = 0.$$

This means that the curvature sphere of principal curvature  $k_i$  is a hyperplane in  $R^{n+1}$ .

### 3 Examples of Möbius deformable hypersurfaces

This section describes the construction of Möbius deformable hypersurfaces  $M^n$  whose highest multiplicity of principal curvatures is  $n - 2$ .

**Example 3.1.** Let  $u : L^m \rightarrow R^{m+1}$  be an immersed hypersurface. We define the cylinder over  $u$  in  $R^{n+1}$  as

$$f = (u, id) : L^m \times R^{n-m} \rightarrow R^{m+1} \times R^{n-m} = R^{n+1},$$

where  $id : R^{n-m} \rightarrow R^{n-m}$  is the identity map.

**Proposition 3.2.** Let  $u, \bar{u} : L^2 \rightarrow R^3$  be a Bonnet pair. Then the cylinders  $f = (u, id) : L^2 \times R^{n-2} \rightarrow R^{n+1}$  and  $\bar{f} = (\bar{u}, id)$  are Möbius deformations to each other.

*Proof.* Let  $\eta$  be the unit normal vector of surface  $u$ . Then  $e_{n+1} = (\eta, \vec{0}) \in R^{n+1}$  is the unit normal vector of hypersurface  $f$ . The first fundamental form  $I$  and the second fundamental form  $II$  of hypersurface  $f$  are given by

$$(8) \quad I = I_u + I_{R^{n-2}}, \quad II = II_u,$$

where  $I_u, II_u$  are the first and second fundamental forms of  $u$ , respectively, and  $I_{R^{n-2}}$  denotes the standard metric of  $R^{n-2}$ . Let  $k_1, k_2$  be principal curvatures of surface  $u$ . The principal curvatures of hypersurface  $f$  are obviously

$$k_1, k_2, 0, \dots, 0.$$

The Möbius metric  $g$  of hypersurface  $f$  is

$$(9) \quad g = \rho^2 I = \frac{n}{n-1} (|II|^2 - nH^2) I = \left( 4H_u^2 - \frac{2n}{n-1} K_u \right) (I_u + I_{R^{n-2}}),$$

where  $H_u, K_u$  are the mean curvature of  $u$  and Gauss curvature of  $u$ , respectively. Since  $\bar{u} : L^2 \rightarrow R^3$  share the same metric  $I_u$  and mean curvature  $H_u$  as  $u$ , the cylinder  $\bar{f} = (\bar{u}, id) : L^2 \times R^{n-2} \rightarrow R^{n+1}$  share the same factor  $\rho$  and Möbius metric, i.e.

$$g = \bar{g}.$$

Note that the correspondence between the Bonnet pair  $u, \bar{u}$  preserves the principal curvatures, yet NOT the principal directions. By (8) this is also true between  $f, \bar{f}$ . So we conclude that  $\bar{f}$  is a non-trivial Möbius deformation to  $f$ . This completes the proof to Proposition 3.2.  $\square$

**Example 3.3.** Let  $u : L^m \longrightarrow S^{m+1} \subset R^{m+2}$  be an immersed hypersurface. We define the cone over  $u$  in  $R^{n+1}$  as

$$f : L^m \times R^+ \times R^{n-m-1} \longrightarrow R^{n+1},$$

$$f(u, t, y) = (tu, y),$$

**Proposition 3.4.** Let  $u, \bar{u} : L^2 \longrightarrow S^3$  be a Bonnet pair in the standard 3-sphere. Then the cone hypersurfaces  $f : L^2 \times R^+ \times R^{n-3} \longrightarrow R^{n+1}$  and  $\bar{f}$  over them are Möbius deformations to each other.

*Proof.* The first and second fundamental forms of hypersurface  $f$  are, respectively,

$$I = t^2 I_u + I_{R^{n-2}}, \quad II = t II_u,$$

where  $I_u, II_u, I_{R^{n-2}}$  are understood as before. Let  $k_1, k_2$  be principal curvatures of surface  $u$ . The principal curvatures of hypersurface  $f$  are

$$\frac{1}{t}k_1, \frac{1}{t}k_2, 0, \dots, 0.$$

Thus the Möbius metric  $g$  of hypersurface  $f$  is

$$(10) \quad g = \rho^2 I = \frac{1}{t^2} \left[ 4H_u^2 - \frac{2n}{n-1}(K_u - 1) \right] (t^2 I_u + I_{R^{n-2}})$$

$$= \left[ 4H_u^2 - \frac{2n}{n-1}(K_u - 1) \right] (I_u + I_{H^{n-2}}),$$

where  $H_u, K_u$  are the mean curvature and Gauss curvature of  $u$ , respectively,  $I_{H^{n-2}}$  is the standard hyperbolic of  $R_+^{n-2} = R^+ \times R^{n-3}$ . Since  $\bar{u} : L^2 \longrightarrow S^3$  share the same metric  $I_u$  and mean curvature  $H_u$  as  $u$ , the cone over  $\bar{u}$   $\bar{f} : L^2 \times R^+ \times R^{n-3} \longrightarrow R^{n+1}$  share the same Möbius metric, i.e.

$$g = \bar{g}.$$

By the same reason in the proof to Proposition 3.2, we know that their principal directions do NOT correspond. So they are genuine deformations to each other. This completes the proof to Proposition 3.4.  $\square$

**Example 3.5.** Let  $R_+^{m+1} = \{(x_1, \dots, x_m, x_{m+1}) \in R^{m+1} | x_{m+1} > 0\}$  be the upper half-space endowed with the standard hyperbolic metric

$$ds^2 = \frac{1}{x_{m+1}^2} \sum_{i=1}^m dx_i^2.$$

Let  $u = (x_1, \dots, x_{m+1}) : M^m \longrightarrow R_+^{m+1}$  be an immersed hypersurface. We define rotational hypersurface over  $u$  in  $R^{n+1}$  as

$$\begin{aligned} f : L^m \times S^{n-m} &\longrightarrow R^{n+1}, \\ f(x_1, \dots, x_{m+1}, \phi) &= (x_1, \dots, x_m, x_{m+1}\phi), \end{aligned}$$

where  $\phi : S^{n-m} \longrightarrow R^{n-m+1}$  is the standard sphere.

**Proposition 3.6.** *Let  $u, \bar{u} : L^2 \longrightarrow R_+^3$  be a Bonnet pair in the hyperbolic 3-space. Then the rotational hypersurfaces  $f = (x_1, x_2, x_3\phi) : L^2 \times S^{n-2} \longrightarrow R^{n+1}$  and  $\bar{f} = (\bar{x}_1, \bar{x}_2, \bar{x}_3\phi)$  are Möbius deformations to each other.*

*Proof.* Let  $R_1^4$  be the Lorentz space with inner product

$$\langle y, y \rangle = -y_1^2 + y_2^2 + y_3^2 + y_4^2, \quad y = (y_1, y_2, y_3, y_4).$$

Let  $H^3 = \{y \in R_1^4 \mid \langle y, y \rangle = -1, y_1 > 0\}$  be the hyperbolic space. Introduce isometry  $\tau : R_+^3 \longrightarrow H^3$  as below:

$$\tau(x_1, x_2, x_3) = \left( \frac{1 + x_1^2 + x_2^2 + x_3^2}{2x_3}, \frac{1 - x_1^2 - x_2^2 - x_3^2}{2x_3}, \frac{x_1}{x_3}, \frac{x_2}{x_3} \right).$$

The inverse  $\tau^{-1} : H^3 \longrightarrow R_+^3$  is  $\tau^{-1}(y_1, y_2, y_3, y_4) = (\frac{y_3}{y_1+y_2}, \frac{y_4}{y_1+y_2}, \frac{1}{y_1+y_2})$ .

Let  $\eta$  be the unit normal vector of surface  $u$  in  $R_+^3$ . Write  $\eta = (\eta_1, \eta_2, \eta_3)$ . Since  $\eta$  is the unit normal vector, then

$$\frac{\eta_1^2 + \eta_2^2 + \eta_3^2}{x_3^2} = 1.$$

Thus the unit normal vector of hypersurface  $f$  in  $R^{n+1}$  is

$$\xi = \frac{1}{x_3}(\eta_1, \eta_2, \eta_3\phi).$$

The first fundamental form of  $u$  is

$$I_u = \frac{1}{x_3^2}(dx_1 \cdot dx_1 + dx_2 \cdot dx_2 + dx_3 \cdot dx_3).$$

The second fundamental form of  $u$  is

$$II_u = -\langle \tau_*(du), \tau_*(d\eta) \rangle = \frac{1}{x_3^2}(dx_1 \cdot d\eta_1 + dx_2 \cdot d\eta_2 + dx_3 \cdot d\eta_3) - \frac{\eta_3}{x_3}I_u.$$

Now we can write out the first and the second fundamental forms of  $f$ :

$$I = dx \cdot dx = x_3^2(I_u + I_{S^{n-2}}), \quad II = x_3 II_u - \eta_3 I_u - \eta_3 I_{S^{n-2}},$$

where  $I_{S^{n-2}}$  is the standard metric of  $S^{n-2}$ . Let  $k_1, k_2$  be principal curvatures of  $u$ . Then principal curvatures of hypersurface  $f$  are

$$\frac{k_1}{x_3} - \frac{\eta_3}{x_3^2}, \frac{k_2}{x_3} - \frac{\eta_3}{x_3^2}, \frac{-\eta_3}{x_3^2}, \dots, \frac{-\eta_3}{x_3^2}.$$

Thus

$$\rho^2 = \frac{n}{n-1}(|II|^2 - nH^2) = \frac{1}{x_3^2} \left[ 4H_u^2 - \frac{2n}{n-1}(K_u + 1) \right],$$

where  $H_u, K_u$  are the mean curvature and Gauss curvature of  $u$ , respectively. So the Möbius metric of hypersurface  $f$  is

$$(11) \quad g = \rho^2 I = \left[ 4H_u^2 - \frac{2n}{n-1}(K_u + 1) \right] (I_u + I_{S^{n-2}}).$$

Since  $u$  and  $\bar{u}$  are a pair of Bonnet surfaces,  $H_u = H_{\bar{u}}, K_u = K_{\bar{u}}, I_u = I_{\bar{u}}$ , thus  $\bar{f} = (\bar{x}_1, \bar{x}_2, \bar{x}_3\phi) : L^2 \times S^{n-2} \rightarrow R^{n+1}$ , the rotational hypersurface over  $\bar{u}$ , is endowed with the same Möbius metric  $g$ . Similar to previous discussions we know that they are NOT congruent. This completes the proof to Proposition 3.6.  $\square$

**Remark 3.7.** We note that the Möbius metric  $g$  in these three cases (9)(10)(11) could be unified in a single formula:

$$(12) \quad g = \left[ 4H_u^2 - \frac{2n}{n-1}(K_u + c) \right] (I_u + I_{N^{n-2}(c)}).$$

Here  $H_u, K_u, I_u$  are the mean curvature, the Gauss curvature and the first fundamental form of the surface  $u : L^2 \rightarrow N^3(-c)$  in a three dimensional space form of constant curvature  $-c$ ;  $I_{N^{n-2}(c)}$  is the Riemannian metric of a  $(n-2)$ -dimensional space form of constant curvature  $c$ . This will be used in Section 9 to show that any Möbius deformation to any example in these three propositions arises in this way. In other words, the possible deformations are as many as that of the corresponding Bonnet surface.

## 4 Hypersurfaces with constant Möbius curvature: deformations and classification

As pointed out in the introduction, hypersurfaces with constant Möbius sectional curvature form a new class of deformable hypersurfaces. In this section, we list hypersurfaces with constant Möbius curvature, i.e., constant sectional curvature with respect to the Möbius metric  $g$ , and compute the Möbius invariants. Then we give a new proof to the classification of such hypersurfaces using a reduction theorem 5.3 in Section 5.

**Example 4.1.** The cylinder in  $R^{n+1}$  over  $\gamma(s) \subset R^2$  is defined by

$$f(s, id) = (\gamma(s), id) : I \times R^{n-1} \longrightarrow R^{n+1} ,$$

where  $id : R^{n-1} \longrightarrow R^{n-1}$  is the identity mapping.

**Remark 4.2.** This is exactly Example 3.1 when  $m = 1$ .

The first fundamental form  $I$  and the second fundamental form  $II$  of hypersurface  $f$  are, respectively,

$$I = ds^2 + I_{R^{n-1}}, \quad II = \kappa(s)ds^2 ,$$

where  $\kappa(s)$  is the geodesic curvature of  $\gamma \subset R^2$ ,  $s$  is the arc-length parameter, and  $I_{R^{n-1}}$  is the standard Euclidean metric of  $R^{n-1}$ . So we have  $(h_{ij}) = \text{diag}(\kappa, 0, \dots, 0)$ ,  $H = \frac{\kappa}{n}$ ,  $\rho = \kappa$ . Thus the Möbius metric  $g$  of hypersurface  $f$  is

$$g = \rho^2 I = \kappa(s)^2 (ds^2 + I_{R^{n-1}}).$$

The Möbius invariants of  $f$  under an orthonormal frame (consisting of principal directions) can be obtained as below using (7):

$$(13) \quad \begin{aligned} C_1 &= -\frac{\kappa_s}{\kappa^2}, \quad C_2 = \dots = C_n = 0, \\ (B_{ij}) &= \text{diag} \left( \frac{n-1}{n}, \frac{-1}{n}, \dots, \frac{-1}{n} \right), \\ (A_{ij}) &= \text{diag}(a_1, a_2, \dots, a_2), \end{aligned}$$

$$\text{where } a_1 = -\frac{\kappa_{ss}}{\kappa^3} + \frac{3}{2} \frac{(\kappa_s)^2}{\kappa^4} + \frac{2n-1}{2n^2}, \quad a_2 = -\frac{1}{2} \left[ \frac{(\kappa_s)^2}{\kappa^4} + \frac{1}{n^2} \right].$$

**Example 4.3.** The cone in  $R^{n+1}$  over  $\gamma(s) \subset S^2(1) \subset R^3$  is defined by

$$f(s, t, id) = (t\gamma(s), id) : I \times R^+ \times R^{n-2} \longrightarrow R^{n+1} ,$$

where  $id : R^{n-2} \longrightarrow R^{n-2}$  is identity mapping and  $R^+ = \{t \mid t > 0\}$ .

**Remark 4.4.** This is exactly Example 3.3 when  $m = 1$ .

The first and second fundamental forms of hypersurface  $f$  are

$$I = t^2 ds^2 + I_{R^{n-1}}, \quad II = t\kappa(s)ds^2 .$$

So we have  $(h_{ij}) = \text{diag}(\frac{\kappa}{t}, 0, \dots, 0)$ ,  $H = \frac{\kappa}{nt}$ ,  $\rho = \frac{\kappa}{t}$ . Thus the Möbius metric  $g$  of hypersurface  $f$  is

$$g = \rho^2 I = \frac{\kappa(s)^2}{t^2} (t^2 ds^2 + I_{R^{n-1}}) = \kappa(s)^2 (ds^2 + I_{R^{n-1}}) ,$$

where  $I_{H^{n-1}}$  is the standard hyperbolic metric of  $H^{n-1}(-1)$ . The Möbius invariants of  $f$  under an orthonormal frame (consisting of principal directions) can be obtained similarly:

$$(14) \quad \begin{aligned} C_1 &= -\frac{\kappa_s}{\kappa^2}, \quad C_2 = \cdots = C_n = 0, \\ (B_{ij}) &= \text{diag} \left( \frac{n-1}{n}, \frac{-1}{n}, \dots, \frac{-1}{n} \right), \\ (A_{ij}) &= \text{diag}(a_1, a_2, \dots, a_2), \end{aligned}$$

$$\text{where } a_1 = -\frac{\kappa_{ss}}{\kappa^3} + \frac{3}{2} \frac{(\kappa_s)^2}{\kappa^4} + \frac{1}{2\kappa^2} + \frac{2n-1}{2n^2}, \quad a_2 = -\frac{1}{2} \left[ \frac{(\kappa_s)^2}{\kappa^4} + \frac{1}{\kappa^2} + \frac{1}{n^2} \right].$$

**Example 4.5.** The rotational hypersurface in  $R^{n+1}$  over  $\gamma(s) \subset R_+^2 = \{(x, y) \in R^2 \mid y > 0\} \subset R^3$  is defined by

$$f(x, y, \theta) = (x, y\theta) : I \times S^{n-1} \longrightarrow R^{n+1},$$

where  $\theta : S^{n-1} \longrightarrow R^n$  is the standard immersion of a round sphere,  $R_+^2$  is regarded as the Poincare half plane with the hyperbolic metric  $ds^2 = \frac{1}{y^2}(dx^2 + dy^2)$ .

**Remark 4.6.** This is exactly Example 3.5 when  $m = 1$ .

In the Poincare half plane, denote the covariant differentiation of the hyperbolic metric as  $D$ . Choose orthonormal frames  $e_1 = y \frac{\partial}{\partial x}, e_2 = y \frac{\partial}{\partial y}$ . It is easy to find

$$D_{e_1}e_1 = e_2, \quad D_{e_1}e_2 = -e_1, \quad D_{e_2}e_1 = D_{e_2}e_2 = 0.$$

For  $\gamma(s) = ((x(s), y(s)) \subset R_+^2$  let  $x'$  denote derivative  $\partial x / \partial s$  and so on. Choose the unit tangent vector  $\alpha = \frac{1}{y}(x'(s)e_1 + y'(s)e_2)$  and the unit normal vector  $\beta = \frac{1}{y}(-y'(s)e_1 + x'(s)e_2)$ . The geodesic curvature is computed via

$$\kappa(s) = \langle D_\alpha \alpha, \beta \rangle = \frac{x'y'' - x''y'}{y^2} + \frac{x'}{y}.$$

After these preparation, we see that the rotational hypersurface  $f(x, y, \theta) = (x, y\theta)$  has differential  $df = (x'ds, y'\theta ds + yd\theta)$  and unit normal vector  $\eta = \frac{1}{y}(-y', x'\theta)$ . Thus the first and second fundamental forms of hypersurface  $f$  are

$$I = df \cdot df = y^2(ds^2 + I_{S^{n-1}}), \quad II = -df \cdot d\eta = (y\kappa - x')ds^2 - x'I_{S^{n-1}},$$

where  $I_{S^{n-1}}$  is the standard metric of  $S^{n-1}(1)$ . Thus principal curvatures are  $\frac{\kappa y - x'}{y^2}, \frac{-x'}{y^2}, \dots, \frac{-x'}{y^2}$ . So  $\rho = \frac{\kappa}{y}$ , and the Möbius metric of  $f$  is

$$g = \rho^2 I = \kappa^2(ds^2 + I_{S^{n-1}}).$$

The coefficients of Möbius invariants are:

$$\begin{aligned}
(15) \quad & C_1 = -\frac{\kappa_s}{\kappa^2}, \quad C_2 = \cdots = C_n = 0, \\
& (B_{ij}) = \text{diag} \left( \frac{n-1}{n}, \frac{-1}{n}, \dots, \frac{-1}{n} \right), \\
& (A_{ij}) = \text{diag}(a_1, a_2, \dots, a_2),
\end{aligned}$$

$$\text{where } a_1 = \frac{\kappa_{ss}}{\kappa^3} - \frac{5}{2} \frac{(\kappa_s)^2}{\kappa^4} - \frac{1}{2\kappa^2} + \frac{2n-1}{2n^2}, \quad a_2 = -\frac{1}{2} \left[ \frac{(\kappa_s)^2}{\kappa^4} - \frac{1}{\kappa^2} + \frac{1}{n^2} \right].$$

**Lemma 4.7.** *The Möbius metric of those hypersurfaces in Examples (4.1), (4.3) and (4.5) are of the warped-product form*

$$(16) \quad g = \kappa^2(s) (ds^2 + I_{-\epsilon}^{n-1}),$$

where  $I_{-\epsilon}^{n-1}$  is the metric of  $n-1$  dimensional space form of constant curvature  $-\epsilon$ . This metric (16) is of constant sectional curvature  $c$  if, and only if, the function  $\kappa(s)$  satisfies

$$(17) \quad \left[ \frac{d}{ds} \frac{1}{\kappa} \right]^2 + \epsilon \left[ \frac{1}{\kappa} \right]^2 = -c.$$

The proof is an easy exercise and we omit it at here.

**Definition 4.8.** *We call a curve  $\gamma$  the curvature-spiral in a 2-dimensional space form  $N^2(\epsilon) = R^2, S^2, H^2$  (of Gauss curvature  $\epsilon = 0, 1, -1$  respectively), if its geodesic curvature  $\kappa(s)$  is not constant and satisfies (17).*

Note that (17) is equivalent to the harmonic oscillator equation for the function  $\kappa(s)$ :

$$(1/\kappa)'' + \epsilon/\kappa = 0.$$

It is easy to see that for fixed  $\epsilon, c$  the solution curve is unique (because  $N^2(\epsilon)$  is a two-point homogeneous space). In particular, when  $\epsilon = 0$ ,  $N^2(\epsilon) = R^2$ , the corresponding  $\gamma$  is a circle or a logarithmic spiral, and the cylinder  $\gamma \times R^{n-1}$  is called *the circular cylinder* and *the spiral cylinder* [23], respectively.

**Theorem 4.9** ([13]). *Let  $f : M^n \rightarrow R^{n+1}$  ( $n \geq 3$ ) be a hypersurface with constant Möbius curvature  $c$ . If  $n = 3$  we assume that  $f$  has two distinct principal curvatures. Then locally  $f$  is Möbius equivalent to one of the following examples:*

- (i) *the circular cylinder (where  $c = 0$ ) or the spiral cylinder (where  $c < 0$ );*
- (ii) *a cone over a curvature-spiral in a 2-sphere (where  $c < 0$ );*
- (iii) *a rotation hypersurface over a curvature-spiral in a hyperbolic 2-plane (the constant curvature  $c$  could be positive, negative or zero).*



*Proof.* Choose an orthonormal frame with respect to  $g$  so that  $(B_{ij})$  is diagonal. According to the remark above,  $f$  has two distinct principal curvatures, one of which is simple. The assumption of constant curvature for  $g$  implies the Ricci curvature  $R_{ij} = 0$  for  $i \neq j$ . From the integrability equation (5) we deduce that  $(A_{ij})$  is also diagonal. Thus the second reduction theorem 5.3 in the next section says that the Möbius form is closed and  $f$  is reducible. Invoking Lemma 4.7 we finish the proof.  $\square$

**Remark 4.10.** *Clearly hypersurfaces with constant Möbius curvature is conformally flat. Equivalently, when the dimension  $n \geq 4$  there must be a principle curvature of multiplicity  $n - 1$  everywhere (and the hypersurface is the envelop of a one-parameter family of  $(n - 1)$  dimensional spheres).*

*On the other hand, a 3-dimensional hypersurface  $f : M^3 \rightarrow R^4$  with constant Möbius sectional curvature may have three distinct principal curvatures. We have finished a classification of such examples which will be published later [20].*

Let's see for fixed  $c$  how many different (global) examples exist. If  $\epsilon = 0$ ,  $N^2(\epsilon) = R^2$ , without loss of generality the solution to (17) is written as

$$(18) \quad \kappa = 1/(\sqrt{-cs}). \quad (\text{logarithmic-spiral})$$

When  $\epsilon = 1$ ,  $N^2(\epsilon) = S^2$ , without loss of generality the solution to (17) is written as

$$(19) \quad \kappa = 1/(\sqrt{-c} \sin s). \quad (\text{sin-spiral})$$

When  $\epsilon = -1$ ,  $N^2(\epsilon) = H^2(-1)$ , there are three different possibilities:

$$(20) \quad \kappa = 1/(\sqrt{-c} \sinh s), \quad (\text{sinh-spiral})$$

$$(21) \quad \kappa = 1/(\sqrt{c} \cosh s), \quad (\text{cosh-spiral})$$

$$(22) \quad \kappa = e^s. \quad (\text{exp-spiral})$$

When  $c > 0$  we have a unique example (cosh-spiral). Yet this example is not homogeneous and should not be viewed as Möbius rigid according to Remark 1.8.

In contrast, for hypersurfaces of Möbius curvature  $c < 0$  we have three non-congruent hypersurfaces: the spiral cylinder, the cone hypersurface, and the rotational hypersurface over the sinh-spiral. We conclude that either of them (in particular, the spiral cylinder) is Möbius deformable. (See Remark 1.7 and 1.8.)

When  $c = 0$ , according to our theorem, there exist two non-congruent examples: the circular cylinder and the rotational hypersurfaces over the exp-spiral as in equation (22). So either of them is deformable.

## 5 The Reduction Theorem

In this section we establish a criterion in terms of Möbius invariants for a hypersurface to be cylinders, cones and rotational hypersurfaces (Examples (3.1)(3.3)(3.5)). This is used in the previous and the final section.

**Theorem 5.1** (Reduction Theorem). *Let  $f : M^n \rightarrow R^{n+1}$  ( $n \geq 3$ ) be a hypersurface. We diagonalize the Möbius second fundamental form under an orthonormal frame  $\{E_1, E_2, \dots, E_n\}$  with respect to the Möbius metric  $g$ :*

$$B_{ij} = \text{diag}\{\lambda_1, \dots, \lambda_m, \mu, \dots, \mu\}.$$

*Assume:*

- (1)  $\lambda_1, \dots, \lambda_m$  are distinct from  $\mu$ .
- (2)  $2 \leq m \leq n - 2$ . (So the multiplicity of  $\mu$  is  $n - m$  and  $2 \leq n - m \leq n - 2$ .)
- (3)  $B_{pq,\alpha} = 0$ ,  $C_\alpha = 0$ ,  $1 \leq p, q \leq m$ ,  $m + 1 \leq \alpha \leq n$ .

*Then  $f$  is Möbius congruent to one of the examples (3.1), (3.3) and (3.5).*

*Proof.* Let  $\{Y, N, Y_1, \dots, Y_n, \xi\}$  be a moving frame in  $R_1^{n+3}$  (see Section 2). In the proof below we adopt the convention on the range of indices as below:

$$1 \leq p, q, r, s, t \leq m, \quad m + 1 \leq \alpha, \beta, \gamma \leq n, \quad 1 \leq i, j, k, l \leq n.$$

Without loss of generality we make a new choice of frame vectors such that

$$(23) \quad A_{\alpha\beta} = a_\alpha \delta_{\alpha\beta}.$$

Applying  $dB_{ij} + \sum_k B_{kj}\omega_{ki} + \sum_k B_{ik}\omega_{kj} = \sum_k B_{ij,k}\omega_k$  for off-diagonal element  $B_{\alpha\beta}$  ( $\alpha \neq \beta$ ) and using the fact  $B_{\alpha\alpha} = B_{\beta\beta} = \mu$ ,  $B_{\alpha\beta} = 0$  we get

$$(24) \quad B_{\alpha\beta,k} = 0 = B_{k\alpha,\beta}, \quad \forall \alpha \neq \beta, \quad 1 \leq k \leq n.$$

The second equality is by the integrability equation. Since  $n - m \geq 2$ , we can always choose indices  $\alpha \neq \beta$ . Then by integrability equation and the assumption  $C_\beta = 0$  one has

$$(25) \quad E_\beta(\mu) = B_{\alpha\alpha,\beta} = B_{\alpha\beta,\alpha} + \delta_{\alpha\alpha}C_\beta - \delta_{\alpha\beta}C_\alpha = C_\beta = 0, \quad \forall \beta.$$

Here  $B_{\alpha\beta,\alpha} = 0$  due to (24). Similarly we have  $B_{p\alpha,q} = B_{pq,\alpha} + \delta_{p\alpha}C_q - \delta_{pq}C_\alpha = B_{pq,\alpha}$  and  $B_{p\alpha,\alpha} = B_{\alpha\alpha,p} - C_p = E_p(\mu) - C_p$ . Together with the assumption  $B_{pq,\alpha} = 0$  we summarize that

$$(26) \quad B_{pq,\alpha} = B_{p\alpha,q} = 0, \quad B_{p\alpha,\alpha} = E_p(\mu) - C_p, \quad \forall p, q, \alpha.$$

Now with the help of (24) and (26) we compute the covariant derivatives of off-diagonal components  $B_{p\alpha}$  and find

$$(27) \quad \omega_{p\alpha} = \frac{B_{p\alpha,\alpha}}{\lambda_p - \mu} \omega_\alpha, \quad \forall p, \alpha.$$

Differentiating once more we obtain the curvature tensor. Compare the coefficient of the component  $\omega_p \wedge \omega_q$  for any given  $p \neq q$ . We find that

$$R_{p\alpha,pq} = 0.$$

(This is the only place where we use the assumption  $m \geq 2$ , to guarantee that there exist such  $p \neq q$ ). From the integrability equation (4) we get

$$(28) \quad A_{q\alpha} = 0, \quad 1 \leq q \leq m, m+1 \leq \alpha \leq n.$$

Similarly by comparing the component  $\omega_p \wedge \omega_\alpha$  we observe that  $R_{p\alpha p\alpha}$  is independent of  $\alpha$  (here we use (26)). Equation (4) yields  $R_{p\alpha p\alpha} = \lambda_p \mu + A_{pp} + A_{\alpha\alpha}$  and

$$(29) \quad A_{\alpha\alpha} = a, \quad \forall \alpha.$$

Next we compute the covariant derivatives of tensor  $A$  and  $C$ . By the condition  $C_\alpha = 0$  and the integrability equation (1)  $A_{ij,k} - A_{ik,j} = B_{ik}C_j - B_{ij}C_k$ ,

$$(30) \quad E_\alpha(a) = E_\alpha(A_{\beta\beta}) = A_{\beta\beta,\alpha} = A_{\alpha\beta,\beta} = 0, \quad \forall \alpha \neq \beta.$$

As a consequence of (27) and  $dC_i + \sum_k C_k \omega_{ki} = \sum_k C_{i,k} \omega_k$  we get that

$$(31) \quad E_\alpha(C_p) = C_{p,\alpha} = C_{\alpha,p} = 0, \quad \forall p, \alpha.$$

Let's look at the geometric meaning of these results. From the formula in (27) we know that distributions

$$D_1 \triangleq \text{Span}\{E_p | 1 \leq p \leq m\}, \quad D_2 \triangleq \text{Span}\{E_\alpha | m+1 \leq \alpha \leq n\}$$

are integrable. Any integral submanifold of distribution  $D_1$  is a  $m$ -dimensional submanifold. On the other hand, along any integral submanifold of  $D_2$  the hypersurface  $Y$  is tangent to

$$(32) \quad F \triangleq \mu Y + \xi,$$

the principal curvature sphere of multiplicity  $n - m$ . Using (25),  $E_p(\mu) = B_{\alpha\alpha,p} = B_{p\alpha,\alpha} + C_p$  and the structure equation it is easy to get that

$$(33) \quad E_\alpha(F) = 0, \quad E_p(F) = B_{p\alpha,\alpha}Y + (\mu - \lambda_p)Y_p.$$

Then principal curvature sphere  $F$  induces a  $m$ -dimensional submanifold in the de-Sitter space  $S_1^{n+2}$

$$F : \widetilde{M}^m = M^n/L \rightarrow S_1^{n+2},$$

where fibers  $L$  are integral submanifolds of distribution  $D_2$ . In other words,  $F$  form a  $m$ -parameter family of  $n$ -spheres enveloped by the hypersurface  $Y$ .

The next crucial observation is that  $F$  is located in a fixed  $(m+2)$ -dimensional linear subspace of  $R_1^{n+3}$ . To show that we compute the repeated derivatives of  $F$ , which contains all information of the envelope  $Y$ . Straightforward yet tedious computation shows that the frames of

$$(34) \quad V_1 \triangleq \text{Span}\{F, E_1(F), \dots, E_m(F), P\}$$

$$\text{where} \quad P \triangleq A_{\alpha\alpha}Y - N + \sum_{p=1}^m \frac{B_{p\alpha,\alpha}}{(\mu - \lambda_p)^2} E_p(F) + \mu F$$

satisfy a linear first order PDE system. Hence these vectors, including  $F$  itself, are contained in a fixed  $(m+2)$ -dimensional subspace  $V_1$  endowed with degenerate, Lorentzian, or positive definite inner product. This agrees with the geometry of cylinders, cones, and rotational hypersurfaces (see examples (3.1),(3.3),(3.5)), where the principal curvature sphere  $F$  is orthogonal to a  $(n-m+1)$ -parameter family of hyperplanes/hyperspheres. Moreover, the orthogonal complement  $V_1^\perp$  of  $\dim = n - m + 1$  contains all  $Y_\alpha, m+1 \leq \alpha \leq n$ .

The final fact above inspires us to proceed in an alternative and easier way. Differentiate any given  $Y_\alpha$  and modulo components in the subspace  $\text{Span}\{Y_\gamma, m+1 \leq \gamma \leq n\}$ . By (23)(28)(27) one finds

$$(35) \quad \begin{aligned} E_i(Y_\alpha) &= -A_{\alpha i}Y - \delta_{\alpha i}N + \sum_j \omega_{\alpha j}(E_i)Y_j + B_{\alpha i}\xi \\ &= \begin{cases} -T \pmod{Y_\gamma}, & \text{when } i = \alpha ; \\ 0 \pmod{Y_\gamma}, & \text{otherwise .} \end{cases} \end{aligned}$$

where

$$(36) \quad T \triangleq A_{\alpha\alpha}Y + N + \sum_{p=1}^m \frac{B_{p\alpha,\alpha}}{\lambda_p - \mu} Y_p - \mu \xi$$

is independent of  $\alpha$  by (26)(29). Then we assert that the subspace

$$(37) \quad V_2 \triangleq \text{Span}\{T, Y_\gamma | m+1 \leq \gamma \leq n\}$$

is parallel along  $M$ . According to our previous computation,  $E_i(Y_\alpha) = 0 \pmod{V_2}, \forall \alpha$ . So we need only to consider  $E_i(T)$ . Fix  $i$  and choose  $\alpha \neq i$ . (Such  $\alpha$  exists by the

assumption  $n - m \geq 2$ , which is the third and final time that we use it. Recall that this condition has been used to derive (25)(30), i.e.  $E_\alpha(\mu) = 0 = E_\alpha(a)$ .) Rewrite the first equality of (35) as

$$(38) \quad T = -E_\alpha(Y_\alpha) + \sum_\gamma \gamma(\cdots) Y_\gamma.$$

By this clever choice of index  $\alpha$  we may prove in a unified way that

$$\begin{aligned} E_i(T) &= -E_i(E_\alpha(Y_\alpha)) + \sum_\gamma \gamma(\cdots) E_i(Y_\gamma) \pmod{Y_\gamma} \\ &= -E_\alpha(E_i(Y_\alpha)) + [E_\alpha, E_i](Y_\alpha) + \sum_\gamma \gamma(\cdots) E_i(Y_\gamma) \pmod{Y_\gamma} \\ &= -E_\alpha(\sum_\beta \gamma(\cdots) Y_\beta) + [E_\alpha, E_i](Y_\alpha) + \sum_\gamma \gamma(\cdots) E_i(Y_\gamma) \pmod{Y_\gamma} \\ &= 0 \pmod{V_2}. \end{aligned}$$

This verifies our previous assertion. More precisely, we have

$$(39) \quad E_p(T) = \frac{B_{p\alpha,\alpha}}{\lambda_p - \mu} T, \quad E_\alpha(T) = Q Y_\alpha, \quad \forall p, \alpha$$

where

$$Q \triangleq \langle T, T \rangle = 2A_{\alpha\alpha} + \mu^2 + \sum_{p=1}^m \frac{B_{p\alpha,\alpha}^2}{(\lambda_p - \mu)^2}$$

satisfies

$$(40) \quad E_p(Q) = \frac{2B_{p\alpha,\alpha}}{\lambda_p - \mu} Q, \quad E_\alpha(Q) = 0.$$

One could verify (39) directly. But the easy way is using  $\langle T, Y_\alpha \rangle = 0$  and (35) to get

$$(41) \quad \langle E_i(T), Y_\alpha \rangle = -\langle T, E_i(Y_\alpha) \rangle = \begin{cases} Q, & \text{when } i = \alpha; \\ 0, & \text{otherwise.} \end{cases}$$

This implies  $E_p(T) \parallel T$  for any  $1 \leq p \leq m$ . Then  $E_p(T)$  as in (39) is derived by differentiating (36) and comparing the  $\xi$  component with  $T$ . The formula for  $E_p(Q)$  in (40) follows directly. On the other hand, we know

$$\langle E_\alpha(T), T \rangle = \frac{1}{2} E_\alpha(Q) = 0,$$

where we used (26) and its consequence  $[E_p, E_\alpha] \in D_2$  together with (25)(30)(31). Combined with (41) we have  $E_\alpha(T) = Q Y_\alpha$ .

Regarding (40) as a linear first-order ODE for  $Q$  we see that  $Q \equiv 0$  or  $Q \neq 0$  on the connected manifold  $M^n$ . Thus there are three possibilities for the induced metric on the fixed subspace  $V_2 \subset \mathbb{R}_1^{n+3}$ .

**Case 1**,  $Q = 0$  on  $M^n$ ;  $V_2$  is endowed with a degenerate inner product.

In this case,  $\langle T, T \rangle = 0$ . By (39),  $E_p(T) \parallel T$ , so  $T$  determines a fixed light-like direction in  $\mathbb{R}_1^{n+3}$ , which we may take to be

$$[T] = [1, -1, 0, \dots, 0] \in \mathbb{R}_1^{n+3}.$$

This corresponds to  $\infty$ , the point at infinity of  $\mathbb{R}^{n+1}$ . Choose space-like vectors  $X_{m+1}, \dots, X_n$  so that  $V_2 = \text{Span}\{T, X_{m+1}, \dots, X_n\}$ . We interpret the geometry of hypersurface  $f : M^n \rightarrow R^{n+1}$  as below:

- 1) Any  $X_\alpha$  determines a hyperplane in  $\mathbb{R}^{n+1}$  because  $\langle T, X_\alpha \rangle = 0$ ;
- 2)  $\text{Span}\{X_\alpha, (m+1 \leq \alpha \leq n)\}$  corresponds to a  $(n-m)$ -dimensional plane  $\Sigma$  in  $\mathbb{R}^{n+1}$ .
- 3)  $F$  is a  $m$ -parameter family of hyperplanes orthogonal to the fixed plane  $\Sigma$ .

$f(M)$ , as the envelope of this family of hyperplanes  $F$ , is clearly a cylinder over a hypersurface  $\widetilde{M} \subset \mathbb{R}^{m+1}$ .

**Case 2**,  $Q < 0$  on  $M^n$ ;  $V_2$  is a Lorentz subspace in  $\mathbb{R}_1^{n+3}$ .

Fix a basis  $\{P_0, P_\infty, X_{m+2}, \dots, X_n\}$  of the  $(n-m+1)$ -dimensional  $V_2$  so that  $P_0, P_\infty$  are light-like. Without loss of generality we may assume

$$P_0 = (1, 1, 0, \dots, 0), \quad P_\infty = (1, -1, 0, \dots, 0).$$

Using the stereographic projection  $\sigma$  they correspond to the origin  $O$  and the point at infinity  $\infty$  of the flat  $\mathbb{R}^{n+1}$ , respectively. We interpret  $F$  and  $V_2$  in terms of the geometry of  $\mathbb{R}^{n+1}$ :

- 1)  $\text{Span}\{X_\alpha : m+2 \leq \alpha \leq n\}$  corresponds to a coordinate plane  $\mathbb{R}^{n-m-1} \subset \mathbb{R}^{n+1}$ , because  $X_\alpha$  must be space-like and orthogonal to  $P_0, P_\infty$ .
- 2)  $F$  is a  $m$ -parameter family of hyperplanes (passing  $O$  and  $\infty$ ) and orthogonal to this fixed  $\mathbb{R}^{n-m-1}$ .

Based on the fact 1),  $f(M)$ , the envelope of  $F$ , is a cylinder over a  $(m+1)$ -dimensional hypersurface in  $\mathbb{R}^{m+2}$  (the orthogonal complement of the previous  $\mathbb{R}^{n-m-1}$ ); moreover, the fact 2) means that  $f(M)$  is a cone (with vertex  $O$ ) over a  $m$ -dimensional hypersurface in  $S^{m+1}$ .

**Case 3**,  $Q > 0$  on  $M^n$ ;  $V_2$  is a space-like subspace.

Without loss of generality we assume that  $P_\infty = (1, -1, 0, \dots, 0)$  is contained in the orthogonal complement of  $V_2$ . As before we make the following interpretation:

- 1)  $V_2$  corresponds to a  $m$ -dimensional plane  $\mathbb{R}^m \subset \mathbb{R}^{n+1}$ .

2)  $F$  is a  $(n - m)$ -parameter family of hyper-spheres orthogonal to this fixed plane  $\mathbb{R}^m$  with centers locating on it. Thus  $F$  envelops a rotational hypersurface  $f(M)$  (over a hypersurface in half-space  $\mathbb{R}_+^{m+1}$ ).

Sum together we complete the proof to the Reduction Theorem.  $\square$

**Remark 5.2.** *It is noteworthy that we may introduce*

$$P \triangleq QY - T$$

which satisfies  $\langle P, T \rangle = 0, \langle P, Y_\alpha \rangle = 0, \langle P, P \rangle = -Q$ . So  $P \perp V_2$  and  $QY = T + P$  is an orthogonal decomposition. Hence a direct proof for case 2 and 3 is to define

$$\bar{P} = \frac{P}{\sqrt{|Q|}}, \quad \theta = \frac{T}{\sqrt{|Q|}}, \quad \langle \bar{P}, \bar{P} \rangle = -\langle \theta, \theta \rangle = \pm 1.$$

Either of them gives a map into the sphere or the hyperbolic space. Then  $M^n = L^m \times N^{n-m}$  is mapped to the lightcone of  $\mathbb{R}_1^{n+3}$  by

$$Y = \frac{-1}{\sqrt{|Q|}}(\bar{P}, \theta) \in \mathbb{R}_1^{n+3} = V_2^\perp \oplus V_2$$

as a warped product of these two maps ( $Q$  depends only on the component of  $\widetilde{M}^m$ ). Clearly such hypersurfaces are cones or rotational hypersurfaces.

The construction of cylinders, cones and rotational hypersurfaces exists for any index  $1 \leq m \leq n - 1$ . From this viewpoint the condition (2) that  $2 \leq n - m \leq n - 2$  in our Reduction Theorem 5.1 is unsatisfying, not only conceptually, but also in that it limits the possible application.

Upon closer examination we find that when  $m = n - 1$  (the Möbius principal curvature  $\mu$  is simple) one could not find a satisfying version of the Reduction Theorem. In particular it seems unavoidable to assume that  $\lambda_1, \dots, \lambda_{n-1}$  (and  $\mu$ ) be distinct (which seems to be a quite unnatural condition), so that we can derive

$$\omega_{pq} = \sum_{r=1}^{n-1} \frac{B_{pq,r}}{\lambda_p - \lambda_q} \omega_r$$

(similar to (27)) and use it to compute  $E_i(T)$ . (As pointed out before (38) in our previous proof of Theorem 5.1, the condition  $m \leq n - 2$  has been used several times, in particular to show  $E_i(T) = 0 \pmod{V_2}$  before (39).) It seems preferable to verify whether the subspace  $V_1$  or  $V_2$  defined in (34)(37) is invariant or not when the Reduction Theorem could not apply directly.

On the other hand, our Reduction Theorem can be generalized to the case  $m = 1$  with some modification on the assumptions.

**Theorem 5.3.** *Let  $f : M^n \rightarrow R^{n+1}$  ( $n \geq 3$ ) be a hypersurface in  $(n+1)$ -dimensional Euclidean space with a principal curvature of multiplicity  $n-1$ . Below are equivalent:*

(1)  *$f$  is Möbius congruent to a cylinder, or a cone, or a rotation hypersurface over a curve  $\gamma \subset N^2(\epsilon)$ .*

(2) *The Möbius form  $\Phi = \sum_i C_i \omega_i$  of  $f$  is closed.*

*Proof.* Write out  $\Phi = \sum_i C_i \omega_i$ , the coefficient matrices  $(B_{ij})$  of the Möbius second fundamental form and  $(A_{ij})$  of the Blaschke tensor under any orthonormal basis  $\{E_1, \dots, E_n\}$  with respect to the Möbius metric  $g$  and dual basis  $\{\omega_1, \dots, \omega_n\}$ . Notice

$$d\Phi = \sum_i dC_i \wedge \omega_i + \sum_i C_i d\omega_i = \sum_{ij} C_{i,j} \omega_j \wedge \omega_i$$

and the integrability equation (2). Then the following are obviously equivalent:

- 1)  $\Phi$  is a closed 1-form;
- 2)  $C_{i,j}$  define a symmetric tensor;
- 3) matrices  $(B_{ij})$  and  $(A_{ij})$  commute;
- 4)  $(B_{ij})$  and  $(A_{ij})$  can be diagonalized simultaneously.

Suppose  $f$  has a principal curvature of multiplicity  $n-1$  and  $\Phi$  is closed. Then we can choose  $\{E_1, \dots, E_n\}$  such that

$$(B_{ij}) = \text{diag}(\lambda, \mu, \dots, \mu), \quad (A_{ij}) = \text{diag}(a_1, a_2, \dots, a_n).$$

We are almost in the same context as in the proof of Theorem 5.1 with  $m=1$  and here we still assume  $1 \leq i, j, k \leq n; 2 \leq \alpha, \beta, \gamma \leq n$ . In particular (24) still holds true and we have  $B_{\alpha\beta, \alpha} = 0$  for any  $\alpha \neq \beta$ .

Using (6) we know  $\lambda = \frac{n-1}{n}, \mu = \frac{-1}{n}$  identically. Differentiate them. We get

$$B_{11, \alpha} = 0, \quad \forall \alpha,$$

$$0 = E_\beta(\mu) = B_{\alpha\alpha, \beta} = B_{\alpha\beta, \alpha} + \delta_{\alpha\alpha} C_\beta - \delta_{\alpha\beta} C_\alpha = C_\beta, \quad \forall \alpha \neq \beta.$$

This looks like (25) and we also use (24)(3). But the assumption is different. Anyway we find that the condition (3) in the Reduction Theorem 5.1 is satisfied. Although here  $m=1$  violates the condition (2), we observe that  $m \geq 2$  is only used only once in that proof to derive (28):

$$A_{q\alpha} = 0,$$

which is an established fact at here already. Thus the previous proof to Theorem 5.1 after (28) is still valid. The same argument shows that  $f$  is reducible.



Conversely, if  $f$  could be reduced to Example (13), (14), or (15), by the computations in the previous section we know that  $(B_{ij})$  and  $(A_{ij})$  can be diagonalized simultaneously, thus  $C$  is closed. This finishes the proof to Theorem 5.3.  $\square$

**Remark 5.4.** In [14], Guo and Lin obtained a classification of hypersurfaces with two distinct principal curvatures and closed Möbius form  $\Phi$ , which included our Theorem 5.3. We give an alternative proof here not only to be self-contained, but also because this proof looks simpler and unified with the Reduction Theorem 5.1.

## 6 Algebraic characteristics of second fundamental forms of deformable hypersurface pairs

Let  $f, \bar{f} : M^n \rightarrow R^{n+1}$  ( $n \geq 4$ ) be two hypersurfaces without umbilics. If they induce the same Möbius metric, i.e.,  $g = \bar{g}$ , then the Möbius second fundamental forms  $B$  of  $f$ , and  $\bar{B}$  of  $\bar{f}$ , have specific algebraic characteristics. The algebraic result is as below:

**Theorem 6.1.** Let  $V$  be a  $n$ -dimensional vector space ( $n \geq 4$ ), and  $B, \bar{B} : V \times V \rightarrow R$  be two bilinear symmetric functions. Let  $\{e_1, \dots, e_n\}$  be an orthonormal basis of  $V$ , and write  $B(e_i, e_j) = B_{ij}$ ,  $\bar{B}(e_i, e_j) = \bar{B}_{ij}$ . Denote

$$(42) \quad \begin{aligned} S_{ijkl} = & B_{ik}B_{jl} - B_{il}B_{jk} \\ & + \frac{1}{n-2} \sum_m \{ \delta_{ik}B_{jm}B_{ml} + \delta_{jl}B_{im}B_{mk} - \delta_{il}B_{jm}B_{mk} - \delta_{jk}B_{im}B_{ml} \}. \end{aligned}$$

Obviously this defines a tensor  $S : V^4 \rightarrow R$  associated with  $B$ .  $\bar{S}$  and  $\bar{S}_{ijkl}$  are defined similarly for  $\bar{B}$ . Assume  $S = \bar{S}$ , i.e.

$$S_{ijkl} = \bar{S}_{ijkl}, \quad \forall 1 \leq i, j, k, l \leq n.$$

Then either  $B$  and  $\bar{B}$  can be diagonalized simultaneously, or there exists an orthonormal basis  $\{e_1, \dots, e_n\}$  of  $V$  such that

$$\{\bar{B}_{ij}\} = \text{diag}(\bar{\lambda}_1, \bar{\lambda}_2, \bar{\mu}, \dots, \bar{\mu}), \{B_{ij}\} = \begin{pmatrix} B_{11} & B_{12} & 0 & \cdots & 0 \\ B_{21} & B_{22} & 0 & \cdots & 0 \\ 0 & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu \end{pmatrix},$$

where  $\bar{\lambda}_1 \neq \bar{\lambda}_2, \mu = \pm \bar{\mu}$ . In the last case there exist an eigenvalue of  $\bar{B}$  with multiplicity at least  $n-2$ .

To prove Theorem 6.1, we need the following two lemmas.

**Lemma 6.2.** *Given  $n \geq 4$ . Assumptions as in Theorem 6.1 except that  $\dim(V) = l, 3 \leq l \leq n$ . That means we still have the fraction  $\frac{1}{n-2}$  in the expression (42), yet the range of those indices is from 1 to  $l$ . Then we can find an orthonormal basis of  $V$  so that  $\{\bar{B}_{ij}\} = \text{diag}(\bar{\lambda}_1, \dots, \bar{\lambda}_l)$  and  $B_{ij} = 0$  for some  $i \neq j$  (i.e. there is at least one off-diagonal element of  $\{B_{ij}\}$  equals to zero).*

*Proof.* Since  $\bar{B}$  is symmetric, we can always diagonalize it as  $\{\bar{B}_{ij}\} = \text{diag}(\bar{\lambda}_1, \dots, \bar{\lambda}_l)$  with respect to an orthonormal basis of  $V$ . If there has been some  $B_{ij} = 0$  with  $i \neq j$  at the same time, we are done. Otherwise, suppose all the off-diagonal elements of  $\{B_{ij}\}$  are non-zero. In this case we make the following

**Assertion:**  $\{\bar{\lambda}_1, \dots, \bar{\lambda}_l\}$  could not be all distinct.

Hence there must exist two equal eigenvalues  $\bar{\lambda}_\alpha = \bar{\lambda}_\beta$ , which enables us to rotate the basis vectors  $\{e_\alpha, e_\beta\}$  properly in the plane  $\text{span}\{e_\alpha, e_\beta\}$  and to obtain a new orthonormal basis of  $V$ , so that  $\{\bar{B}_{ij}\}$  is still a diagonal matrix and  $B_{\alpha\beta} = 0$ . This completes the proof.

To prove the assertion above (on condition that  $B_{ij} \neq 0, \forall i \neq j$ ), we substitute the expressions of  $S, \bar{S}$  and  $\{\bar{B}_{ij}\} = \text{diag}(\bar{\lambda}_1, \dots, \bar{\lambda}_l)$  into the equality

$$S_{\alpha 1 \alpha 1} - S_{\alpha 2 \alpha 2} = \bar{S}_{\alpha 1 \alpha 1} - \bar{S}_{\alpha 2 \alpha 2}, \quad \forall 3 \leq \alpha \leq l.$$

As the result we obtain

$$\begin{aligned} (43) \quad & B_{\alpha\alpha}(B_{11} - B_{22}) - (B_{1\alpha}^2 - B_{2\alpha}^2) + \frac{1}{n-2} \sum_{m=1}^l (B_{1m}^2 - B_{2m}^2) \\ & = (\bar{\lambda}_1 - \bar{\lambda}_2) \left[ \bar{\lambda}_\alpha + \frac{1}{n-2} (\bar{\lambda}_1 + \bar{\lambda}_2) \right], \quad \forall 3 \leq \alpha \leq l. \end{aligned}$$

In the following let the range of the index  $\alpha$  be  $3 \leq \alpha \leq l$ . We want to show that the left hand side of (43) vanishes. First note that  $\{\bar{B}_{ij}\} = \text{diag}(\bar{\lambda}_1, \dots, \bar{\lambda}_l)$  implies

$$\bar{S}_{ijk} = 0$$

when  $i, j, k$  are distinct. It follows from the equality  $S_{ijk} = \bar{S}_{ijk}$  that

$$(44) \quad B_{ii}B_{jk} - B_{ij}B_{ik} + \frac{1}{n-2} \sum_{m=1}^l B_{jm}B_{km} = 0, \quad \forall \text{ distinct } i, j, k.$$

Hence

$$\begin{aligned}
(45) \quad \frac{B_{11}}{B_{12}} - \frac{B_{1\alpha}}{B_{2\alpha}} &= -\frac{1}{n-2} \cdot \frac{1}{B_{12}B_{2\alpha}} \cdot \sum_{m=1}^l B_{2m}B_{m\alpha} \\
&= -\frac{1}{n-2} \left[ \frac{B_{1\alpha}}{B_{2\alpha}} + \frac{B_{22}}{B_{12}} + \sum_{m=3}^l \frac{B_{m\alpha}}{B_{12}} \cdot \frac{B_{2m}}{B_{2\alpha}} \right].
\end{aligned}$$

Similarly one can find

$$\begin{aligned}
(46) \quad \frac{B_{22}}{B_{12}} - \frac{B_{2\alpha}}{B_{1\alpha}} &= -\frac{1}{n-2} \cdot \frac{1}{B_{12}B_{1\alpha}} \cdot \sum_{m=1}^l B_{1m}B_{m\alpha} \\
&= -\frac{1}{n-2} \left[ \frac{B_{2\alpha}}{B_{1\alpha}} + \frac{B_{11}}{B_{12}} + \sum_{m=3}^l \frac{B_{m\alpha}}{B_{12}} \cdot \frac{B_{1m}}{B_{1\alpha}} \right].
\end{aligned}$$

Taking (45) – (46) yields

$$\left[ \frac{B_{11} - B_{22}}{B_{12}} - \frac{B_{1\alpha}}{B_{2\alpha}} + \frac{B_{2\alpha}}{B_{1\alpha}} \right] \left( 1 - \frac{1}{n-2} \right) = -\frac{1}{n-2} \sum_{m=3}^l \frac{B_{m\alpha}}{B_{12}} \left( \frac{B_{2m}}{B_{2\alpha}} - \frac{B_{1m}}{B_{1\alpha}} \right) = 0$$

due to  $B_{2m}B_{1\alpha} - B_{2\alpha}B_{1m} = S_{21m\alpha} = \bar{S}_{21m\alpha} = 0$  when  $\bar{B}$  is diagonal and  $m, \alpha \geq 3$ . We conclude

$$\frac{B_{11} - B_{22}}{B_{12}} = \frac{B_{1\alpha}}{B_{2\alpha}} - \frac{B_{2\alpha}}{B_{1\alpha}} = b$$

for some constant  $b$ . It follows that

$$\begin{aligned}
&B_{\alpha\alpha}(B_{11} - B_{22}) - (B_{1\alpha}^2 - B_{2\alpha}^2) + \frac{1}{n-2} \sum_{m=1}^l (B_{1m}^2 - B_{2m}^2) \\
&= B_{\alpha\alpha}(B_{11} - B_{22}) - (B_{1\alpha}^2 - B_{2\alpha}^2) + \frac{1}{n-2} (B_{11}^2 - B_{22}^2) + \frac{1}{n-2} \sum_{m=3}^l (B_{1m}^2 - B_{2m}^2) \\
&= B_{\alpha\alpha} \cdot bB_{12} - b \cdot B_{1\alpha}B_{2\alpha} + \frac{1}{n-2} (B_{11} + B_{22}) \cdot bB_{12} + \frac{1}{n-2} \sum_{m=3}^l (b \cdot B_{1m}B_{2m}) \\
&= b \left[ B_{\alpha\alpha}B_{12} - B_{1\alpha}B_{2\alpha} + \frac{1}{n-2} \sum_{m=1}^l B_{1m}B_{2m} \right] = 0
\end{aligned}$$

by (44). From (43) we have

$$(47) \quad (\bar{\lambda}_1 - \bar{\lambda}_2) \left[ \bar{\lambda}_\alpha + \frac{1}{n-2} (\bar{\lambda}_1 + \bar{\lambda}_2) \right] = 0.$$

So either  $\bar{\lambda}_1 = \bar{\lambda}_2$ , or  $\bar{\lambda}_\alpha = -\frac{1}{n-2}(\bar{\lambda}_1 + \bar{\lambda}_2)$ ,  $\forall 3 \leq \alpha \leq l$ . This verifies the assertion when  $l \geq 4$ .

The only case unsolved is when  $l = 3$ . This time (47) takes the form

$$(\bar{\lambda}_1 - \bar{\lambda}_2)[\bar{\lambda}_3 + \frac{1}{n-2}(\bar{\lambda}_1 + \bar{\lambda}_2)] = 0.$$

Taking permutation of the indices 1, 2, 3 yields two other similar formulas. Now it is easy to prove that  $\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3$  can not be all distinct by contradiction. Hence the proof to Lemma 6.2 is finished.  $\square$

**Remark 6.3.** Note that in the proof above we used the fact  $\frac{1}{n-2} \neq 1$  at two places. Thus the condition  $n \geq 4$  is necessary. On the other hand, by the integrability equations (4) the Weyl conformal tensor associated with the Möbius metric  $g$  can be expressed by the Möbius invariants as below:

$$\begin{aligned} C_{ijkl} &= B_{ik}B_{jl} - B_{il}B_{jk} - \frac{1}{n(n-2)}(\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il}) \\ &\quad + \frac{1}{n-2} \sum_m \{\delta_{ik}B_{jm}B_{ml} + \delta_{jl}B_{im}B_{mk} - \delta_{il}B_{jm}B_{mk} - \delta_{jk}B_{im}B_{ml}\} \\ &= S_{ijkl} - \frac{1}{n(n-2)}(\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il}). \end{aligned}$$

It is well known that the Weyl conformal tensor vanishes on three dimensional Riemannian manifold. Therefore when  $n = 3$ ,  $S_{ijkl} = \frac{1}{3}(\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il}) = \bar{S}_{ijkl}$  is a trivial identity.

**Lemma 6.4.** Assumptions as in Lemma 6.2. By the conclusion above, without loss of generality we may suppose that for a given orthonormal basis of  $V$  there are  $\{\bar{B}_{ij}\} = \text{diag}(\bar{\lambda}_1, \dots, \bar{\lambda}_l)$  and  $B_{ij} = 0$  for some  $i \neq j$ . Then there exists a properly chosen new orthonormal basis of  $V$ , with respect to which  $\{\bar{B}_{ij}\}$  is still diagonal and

$$\{B_{ij}\} = \begin{pmatrix} B_{11} & \cdots & B_{1,l-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ B_{l-1,1} & \cdots & B_{l-1,l-1} & 0 \\ 0 & \cdots & 0 & B_{ll} \end{pmatrix}$$

is a semi-diagonal matrix.

*Proof.* For simplicity denote  $k = l - 1$ . Without loss of generality we may assume that the off-diagonal element  $B_{kl} = 0$ .

First we consider the easy case  $l = 3$ . As in (44), we have

$$0 = B_{11}B_{23} - B_{12}B_{13} + \frac{1}{n-2} \sum_{m=1}^3 B_{2m}B_{m3} = \left( \frac{1}{n-2} - 1 \right) B_{12}B_{13}$$

because  $B_{23} = 0$  as assumed. It follows that either  $B_{12} = 0$  or  $B_{13} = 0$ , and the conclusion is proved.

In general, when  $l \geq 4$ , for any  $i < j < k = l - 1$  there is

$$(48) \quad 0 = \bar{S}_{ikjl} = S_{ikjl} = B_{ij}B_{kl} - B_{il}B_{kj} = -B_{il}B_{kj}.$$

If  $B_{il} = 0$  for any  $i < k = l - 1$ , then all the off-diagonal elements in the  $l$ -th column and the  $l$ -th row vanish, and we are done. Otherwise, suppose  $B_{1l} \neq 0$  without loss of generality. Then by (48),  $B_{jk} = 0$ ,  $\forall 1 < j < k = l - 1$ . Using this result and  $B_{lk} = 0, B_{1l} \neq 0$ , we may prove  $B_{1k} = 0$  by (44):

$$0 = B_{11}B_{kl} - B_{1k}B_{1l} + \frac{1}{n-2} \sum_{j=1}^l B_{jk}B_{jl} = \left( \frac{1}{n-2} - 1 \right) B_{1k}B_{1l}.$$

So  $B_{jk} = 0$ ,  $\forall j \neq k$ . That means all the off-diagonal elements in the  $k$ -th column and the  $k$ -th row vanish. Interchanging the basis vectors  $e_k$  and  $e_l$  gives the desired result. The proof to Lemma 6.4 is finished.  $\square$

*Proof to Theorem 6.1.* From Lemma 6.2 and Lemma 6.4 and by induction it is easy to see that  $\{B_{ij}\}, \{\bar{B}_{ij}\}$  can be diagonalized simultaneously except that  $B_{12}$  might be non-zero.

Denote  $\{\bar{B}_{ij}\} = \text{diag}(\bar{\lambda}_1, \dots, \bar{\lambda}_n)$  as before. When  $B_{12} = 0$  it is the first case in the conclusion. If  $B_{12} \neq 0$  yet  $\bar{\lambda}_1 = \bar{\lambda}_2$ , one might rotate the basis vectors  $\{e_1, e_2\}$  properly in the plane  $\text{span}\{e_1, e_2\}$  and obtain a new orthonormal basis of  $V$  such that  $\{\bar{B}_{ij}\}$  is invariant and  $B_{12} = 0$ , hence we are also done. The final part of the proof is to show that when  $B_{12} \neq 0$  and  $\bar{\lambda}_1 \neq \bar{\lambda}_2$ ,  $\{B_{ij}\}$  and  $\{\bar{B}_{ij}\}$  must have the desired multiplicities of their eigenvalues.

Again by (44),  $\forall 3 \leq \alpha \leq n$ ,

$$0 = B_{\alpha\alpha}B_{12} - B_{\alpha1}B_{\alpha2} + \frac{1}{n-2} \sum_{m=1}^n B_{1m}B_{m2} = B_{12} \left[ B_{\alpha\alpha} + \frac{1}{n-2}(B_{11} + B_{22}) \right].$$

Thus  $B_{\alpha\alpha} = -\frac{1}{n-2}(B_{11} + B_{22}) = \mu$  for any  $\alpha \geq 3$ . So  $\{B_{ij}\}$  has the desired form. As a by-product we find that

$$(49) \quad \text{tr}(B) = B_{11} + B_{22} + (n-2)\mu = 0.$$

Taking use of the fact above and the equalities  $S_{1\alpha1\alpha} = \bar{S}_{1\alpha1\alpha}, S_{2\alpha2\alpha} = \bar{S}_{2\alpha2\alpha}$ , we have

$$(50) \quad B_{11}\mu + \frac{1}{n-2} (\lambda^2 + B_{11}^2 + B_{12}^2) = \bar{\lambda}_1\bar{\lambda}_\alpha + \frac{1}{n-2} (\bar{\lambda}_\alpha^2 + \bar{\lambda}_1^2),$$

$$(51) \quad B_{22}\mu + \frac{1}{n-2} (\lambda^2 + B_{22}^2 + B_{12}^2) = \bar{\lambda}_2\bar{\lambda}_\alpha + \frac{1}{n-2} (\bar{\lambda}_\alpha^2 + \bar{\lambda}_2^2)$$

for any  $\alpha \geq 3$ . Taking (50) – (51) yields

$$(B_{11} - B_{22}) \left[ B_{\alpha\alpha} + \frac{1}{n-2}(B_{11} + B_{22}) \right] = (\bar{\lambda}_1 - \bar{\lambda}_2) \left[ \bar{\lambda}_\alpha + \frac{1}{n-2}(\bar{\lambda}_1 + \bar{\lambda}_2) \right].$$

The left hand side vanishes by (49). It follows that  $\bar{\lambda}_\alpha = -\frac{1}{n-2}(\bar{\lambda}_1 + \bar{\lambda}_2) = \bar{\lambda}$  for all  $\alpha \geq 3$  (keep in mind that  $\bar{\lambda}_1 \neq \bar{\lambda}_2$  at here) and  $\text{tr}(\bar{B}) = 0$ . Finally  $S_{3434} = \bar{S}_{3434}$  implies  $\mu^2 = \bar{\mu}^2$ . This finishes the proof to Theorem 6.1.  $\square$

## 7 Hypersurfaces with low multiplicities: rigidity

Let  $f, \bar{f} : M^n \rightarrow R^{n+1}$  ( $n \geq 4$ ) be two hypersurfaces without umbilics. In this section and the following two, the Möbius invariants of  $f$  will be denoted by  $\{A, B, C\}$  and those of  $\bar{f}$  by  $\{\bar{A}, \bar{B}, \bar{C}\}$ . Because of the local nature of our results, we can assume that the multiplicities of all principal curvatures are constant in an open, connected subset (which is still denoted by  $M$ ). Deformable cases will be treated in the next two sections.

**Theorem 7.1.** *Let  $f, \bar{f} : M^n \rightarrow R^{n+1}$  ( $n \geq 4$ ) be two hypersurfaces without umbilics. Assume that they induce the same Möbius metrics  $g$ , and all principal curvatures of  $B$  have multiplicity less than  $n - 2$  everywhere. Then  $f$  is Möbius congruent to  $\bar{f}$ .*

We divide our proof into two parts. The case of dimension  $n = 4$  is different from higher dimensional case ( $n \geq 5$ ) and need to be discussed separately. Before that we make some preparation first.

The same Möbius metric  $g$  for  $f, \bar{f}$  determines the same curvature tensor  $R_{ijkl}$ . By the integrability equations (4)(5), the conclusion of Theorem 6.1 applies to the Möbius second fundamental forms  $B, \bar{B}$ . Since the multiplicities of all principal curvatures are less than  $n - 2$  at here by assumption, Theorem 6.1 guarantees that locally we can choose an orthonormal basis  $\{E_1, \dots, E_n\}$  with respect to  $g$  such that

$$\{B_{ij}\} = \text{diag}(\lambda_1, \dots, \lambda_n); \{\bar{B}_{ij}\} = \text{diag}(\bar{\lambda}_1, \dots, \bar{\lambda}_n).$$

Now (4)(5) imply

$$(52) \quad \lambda_i \lambda_j + \frac{1}{n-2}(\lambda_i^2 + \lambda_j^2) = \bar{\lambda}_i \bar{\lambda}_j + \frac{1}{n-2}(\bar{\lambda}_i^2 + \bar{\lambda}_j^2), \quad \forall i \neq j.$$

Changing the subscript of (52) and taking difference, we get

$$(53) \quad (\lambda_i - \lambda_k)[\lambda_j + \frac{1}{n-2}(\lambda_i + \lambda_k)] = (\bar{\lambda}_i - \bar{\lambda}_k)[\bar{\lambda}_j + \frac{1}{n-2}(\bar{\lambda}_i + \bar{\lambda}_k)], \quad \forall \text{ distinct } i, j, k.$$

To obtain the rigidity result we need only to show that  $\bar{B} = \pm B$ ; reverse the direction of the normal vector field of  $f$  if necessary we will have  $\bar{B} = B$ , which shows that  $f$  is congruent to  $\bar{f}$  by the fundamental theorem 2.2.

**Proposition 7.2.** *The conclusion of Theorem 7.1 is valid when the dimension  $n \geq 5$ .*

*Proof.* We assert that there is a linear relation between  $\lambda_j$  and  $\bar{\lambda}_j$ , i.e. there exists constants  $b, c$  such that

$$\bar{\lambda}_j = b\lambda_j + c, \quad \forall 1 \leq j \leq n.$$

In other words, regard  $p_j = (\lambda_j, \bar{\lambda}_j)$  as coordinates of  $n$  points on a plane, then these  $n$  points are collinear.

Without loss of generality assume that  $\lambda_1 \neq \lambda_2$ . We just show  $p_3 = (\lambda_3, \bar{\lambda}_3)$  is collinear with  $p_1 = (\lambda_1, \bar{\lambda}_1), p_2 = (\lambda_2, \bar{\lambda}_2)$ . (For any other index  $j \neq 1, 2, 3$  the proof is the same.) Now we need to consider two cases separately.

In the first case,  $n \geq 5$  and the highest multiplicity of principal curvatures is less than  $n - 3$ . We can find  $\lambda_i \neq \lambda_k$  which are distinct from  $\{1, 2, 3\}$ . Fix  $i, k$  in (53), we see that all other  $(\lambda_j, \bar{\lambda}_j)$  ( $j \neq i, k$ ) satisfies a non-trivial linear equation (53). In particular,  $p_1, p_2, p_3$  are collinear.

In the second case,  $\lambda_i$  might be a constant for any indices  $i \neq 1, 2, 3$ . (Note that  $\lambda_i \neq \lambda_1, \lambda_2, \lambda_3$ . Otherwise there will be a principal curvature of multiplicity at least  $n - 2$ , contradiction. Yet  $\lambda_3$  might be equal to either of  $\lambda_1, \lambda_2$ .) Fix  $i = 1, k = 5$ , we have  $\lambda_1 \neq \lambda_5$  and by (53) we know

$$p_2 = (\lambda_2, \bar{\lambda}_2), p_3 = (\lambda_3, \bar{\lambda}_3), p_4 = (\lambda_4, \bar{\lambda}_4) \text{ are collinear.}$$

Similarly we know  $\{p_1, p_2, p_4\}$  and  $\{p_1, p_3, p_4\}$  are collinear triples. This guarantees that  $\{p_1, p_2, p_3\}$  (and other  $p_j$ 's) are collinear and finishes the proof to our assertion.

Now we know  $\bar{\lambda}_j = b\lambda_j + c$  for constants  $b, c$  and for any  $j$ . The fact  $\sum_j \lambda_j = 0 = \sum_j \bar{\lambda}_j$  (the first identity in (6)) implies  $c = 0$ . Using the second identity  $\sum_j \lambda_j^2 = \frac{n-1}{n} = \sum_j \bar{\lambda}_j^2$  in (6) we conclude that  $b = \pm 1$ . This completes the proof to Proposition 7.2.  $\square$

**Proposition 7.3.** *The conclusion of Theorem 7.1 is valid when dimension  $n = 4$ .*

*Proof.* First note that when  $n = 4$  and the highest multiplicity is less than  $n - 2 = 2$ , four principal curvatures of  $f$  are distinct. Consider

$$\{B_{ij}\} = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4); \quad \{\bar{B}_{ij}\} = \text{diag}(\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3, \bar{\lambda}_4).$$

By (52) and  $n = 4$  we have

$$(54) \quad \lambda_i + \lambda_j = \pm(\bar{\lambda}_i + \bar{\lambda}_j), \quad i \neq j.$$

We assert that there are four possibilities on each and every point of  $M$ :

- (1)  $B = \pm \bar{B}$ ;
- (2)  $\{B_{ij}\} = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4), \{\bar{B}_{ij}\} = \text{diag}(\pm\lambda_2, \pm\lambda_1, \pm\lambda_4, \pm\lambda_3)$ ;
- (3)  $\{B_{ij}\} = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4), \{\bar{B}_{ij}\} = \text{diag}(\pm\lambda_3, \pm\lambda_4, \pm\lambda_1, \pm\lambda_2)$ ;
- (4)  $\{B_{ij}\} = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4), \{\bar{B}_{ij}\} = \text{diag}(\pm\lambda_4, \pm\lambda_3, \pm\lambda_2, \pm\lambda_1)$ .

Suppose  $B \neq \pm \bar{B}$ . Consider a special case

$$\lambda_1 + \lambda_2 = (\bar{\lambda}_1 + \bar{\lambda}_2), \quad \lambda_1 + \lambda_3 = -(\bar{\lambda}_1 + \bar{\lambda}_3), \quad \lambda_1 + \lambda_4 = -(\bar{\lambda}_2 + \bar{\lambda}_3).$$

Taking sum of these three equalities and using the fact  $\sum_j \lambda_j = 0 = \sum_j \bar{\lambda}_j$  we get  $\lambda_1 = \bar{\lambda}_2$ . Substitute this back and use  $\sum_j \lambda_j = 0 = \sum_j \bar{\lambda}_j$  again. We conclude that this is case (2). Reversing either of the normal vector fields and taking permutations reduce other possibilities to this special case. This verifies our assertion.

We need to exclude possibility (2) by contradiction. Other cases are similar. This time  $\lambda_1 + \lambda_2 = 0$  automatically implies  $B = \pm \bar{B}$  by  $\sum_j \lambda_j = 0 = \sum_j \bar{\lambda}_j$ . So we need only to find contradiction when  $\lambda_1 \neq \pm\lambda_2, \lambda_3 \neq \pm\lambda_4$  and

$$\{B_{ij}\} = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4), \{\bar{B}_{ij}\} = \text{diag}(\pm\lambda_2, \pm\lambda_1, \pm\lambda_4, \pm\lambda_3)$$

under a locally orthonormal basis  $\{E_1, \dots, E_4\}$  with respect to  $g$ .

Using the covariant derivative of  $B$  and  $\bar{B}$ , we get

$$(55) \quad (\lambda_i - \lambda_j)\omega_{ij} = \sum_k B_{ij,k}\omega_k, \quad (\bar{\lambda}_i - \bar{\lambda}_j)\omega_{ij} = \sum_k \bar{B}_{ij,k}\omega_k, \quad i \neq j.$$

So

$$(56) \quad (\bar{\lambda}_i - \bar{\lambda}_j)B_{ij,k} = (\lambda_i - \lambda_j)\bar{B}_{ij,k}, \quad i \neq j.$$

Consequently, there is

$$(57) \quad B_{12,k} = -\bar{B}_{12,k}, \quad B_{34,k} = -\bar{B}_{34,k},$$

because  $\bar{\lambda}_1 - \bar{\lambda}_2 = \lambda_2 - \lambda_1 \neq 0, \bar{\lambda}_3 - \bar{\lambda}_4 = \lambda_4 - \lambda_3 \neq 0$ . It follows that

$$(58) \quad B_{ij,k} = \bar{B}_{ij,k} = 0, \quad \text{when } i, j, k \text{ are distinct.}$$



To verify (58), consider the case when  $i = 1, j = 2, k = 3$ . If  $B_{12,3} \neq 0$ , then  $B_{13,2} = -\bar{B}_{13,2} \neq 0$ . (Note  $B_{ij,k} = B_{ik,j}$  when  $i, j, k$  are distinct by (3).) Combined with (56), there should be  $\lambda_1 - \lambda_3 = \bar{\lambda}_3 - \bar{\lambda}_1 = \lambda_4 - \lambda_2$ . Yet this implies  $\lambda_1 + \lambda_2 = 0$  under our condition  $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$ , which contradicts the assumption  $\lambda_1 \neq \pm\lambda_2$ . Other cases are verified similarly. As a corollary of (55) and (58),

$$(59) \quad (\lambda_i - \lambda_j)\omega_{ij} = B_{ij,i}\omega_i + B_{ij,j}\omega_j, \quad (\bar{\lambda}_i - \bar{\lambda}_j)\omega_{ij} = \bar{B}_{ij,i}\omega_i + \bar{B}_{ij,j}\omega_j, \quad i \neq j.$$

To derive the exact expressions of the connection forms  $\omega_{ij}$ , we shall compute out all the quantities like  $B_{ii,j}$  in terms of  $\lambda_k$ 's and  $C_k$ 's. By the symmetry in our situation, obviously there is  $\bar{B}_{33,1} = B_{44,1}, \bar{B}_{44,1} = B_{33,1}$ . Together with (56) and (3), it follows

$$(\bar{\lambda}_3 - \bar{\lambda}_1)(B_{33,1} - C_1) = (\bar{\lambda}_3 - \bar{\lambda}_1)B_{31,3} = (\lambda_3 - \lambda_1)\bar{B}_{31,3} = (\lambda_3 - \lambda_1)(\bar{B}_{33,1} - \bar{C}_1).$$

So we get

$$(60) \quad \begin{aligned} (\lambda_4 - \lambda_2)(B_{33,1} - C_1) &= (\lambda_3 - \lambda_1)(B_{44,1} - \bar{C}_1), \\ (\lambda_3 - \lambda_2)(B_{44,1} - C_1) &= (\lambda_4 - \lambda_1)(B_{33,1} - \bar{C}_1). \end{aligned}$$

The second equation is obtained in the similar way.

On the other hand, (57) tells us that

$$B_{22,1} - C_1 = B_{12,2} = -\bar{B}_{12,2} = \bar{B}_{22,1} - \bar{C}_1 = B_{11,1} - \bar{C}_1.$$

Thus  $B_{11,1} + B_{22,1} = C_1 + \bar{C}_1$ . Note that  $\sum_i \lambda_i = \sum_i B_{ii} = 0$  and  $\sum_i B_{ii}^2 = \frac{n-1}{n}$  imply

$$(61) \quad \sum_{i=1}^4 B_{ii,k} = 0, \quad \sum_{i=1}^4 \lambda_i B_{ii,k} = 0, \quad \forall k.$$

In particular,

$$(62) \quad B_{33,1} + B_{44,1} = -(B_{11,1} + B_{22,1}) = -C_1 - \bar{C}_1.$$

Eliminating  $B_{33,1}, B_{44,1}$  from (62)(60) (keep in mind  $\sum_i \lambda_i = 0$ ) yields  $\lambda_2 C_1 = \lambda_1 \bar{C}_1$ . Because  $\lambda_1, \lambda_2$  could not be zero at the same time ( $\lambda_1 \neq \pm\lambda_2$ ), we may denote

$$(63) \quad \Delta_1 := \frac{C_1}{\lambda_1} = \frac{\bar{C}_1}{\lambda_2}.$$

In case that  $\lambda_1 = 0 \neq \lambda_2$ , there must be  $C_1 = 0$ , and we need only to take  $\Delta_1 = \frac{\bar{C}_1}{\lambda_2}$  which is well-defined.

Putting (63) into (60)(62) solves  $B_{33,1}, B_{44,1}$ . Then by (61) we get the complete solution:

$$(64) \quad \begin{aligned} B_{11,1} &= \frac{\lambda_2\lambda_3 + \lambda_2\lambda_4 - \lambda_3^2 - \lambda_4^2}{\lambda_1 - \lambda_2} \Delta_1, \quad B_{33,1} = \lambda_3 \Delta_1, \\ B_{22,1} &= \frac{\lambda_1\lambda_3 + \lambda_1\lambda_4 - \lambda_3^2 - \lambda_4^2}{\lambda_2 - \lambda_1} \Delta_1, \quad B_{44,1} = \lambda_4 \Delta_1. \end{aligned}$$

Similarly there are:

$$(65) \quad \begin{aligned} B_{11,2} &= \frac{\lambda_2\lambda_3 + \lambda_2\lambda_4 - \lambda_3^2 - \lambda_4^2}{\lambda_1 - \lambda_2} \Delta_2, \quad B_{33,2} = \lambda_3 \Delta_2, \\ B_{22,2} &= \frac{\lambda_1\lambda_3 + \lambda_1\lambda_4 - \lambda_3^2 - \lambda_4^2}{\lambda_2 - \lambda_1} \Delta_2, \quad B_{44,2} = \lambda_4 \Delta_2, \\ B_{11,3} &= \lambda_1 \Delta_3, \quad B_{33,3} = \frac{\lambda_4\lambda_1 + \lambda_4\lambda_2 - \lambda_1^2 - \lambda_2^2}{\lambda_3 - \lambda_4} \Delta_3, \\ B_{22,3} &= \lambda_2 \Delta_3, \quad B_{44,3} = \frac{\lambda_3\lambda_1 + \lambda_3\lambda_2 - \lambda_1^2 - \lambda_2^2}{\lambda_4 - \lambda_3} \Delta_3, \\ B_{11,4} &= \lambda_1 \Delta_4, \quad B_{33,4} = \frac{\lambda_4\lambda_1 + \lambda_4\lambda_2 - \lambda_1^2 - \lambda_2^2}{\lambda_3 - \lambda_4} \Delta_4, \\ B_{22,4} &= \lambda_2 \Delta_4, \quad B_{44,4} = \frac{\lambda_3\lambda_1 + \lambda_3\lambda_2 - \lambda_1^2 - \lambda_2^2}{\lambda_4 - \lambda_3} \Delta_4, \end{aligned}$$

where

$$\Delta_2 := \frac{C_2}{\lambda_2} = \frac{\bar{C}_2}{\lambda_1}, \quad \Delta_3 := \frac{C_3}{\lambda_3} = \frac{\bar{C}_3}{\lambda_4}, \quad \Delta_4 := \frac{C_4}{\lambda_4} = \frac{\bar{C}_4}{\lambda_3}.$$

Now the connection forms  $\omega_{ij}$  could be determined. Since  $\lambda_1 - \lambda_2 \neq 0$ , by (59)(3) (64)(65) and  $C_1 = \lambda_1 \Delta_1, C_2 = \lambda_2 \Delta_2, \sum_i \lambda_i = 0$ , we get

$$\omega_{12} = \frac{B_{11,2} - C_2}{\lambda_1 - \lambda_2} \omega_1 + \frac{B_{22,1} - C_1}{\lambda_1 - \lambda_2} \omega_2 = I_{12}(\Delta_2 \omega_1 - \Delta_1 \omega_2), \quad I_{12} := -\frac{2\lambda_1\lambda_2 + \lambda_3^2 + \lambda_4^2}{(\lambda_1 - \lambda_2)^2}.$$

Similarly, there is

$$\omega_{34} = I_{34}(\Delta_4 \omega_3 - \Delta_3 \omega_4), \quad I_{34} := -\frac{2\lambda_3\lambda_4 + \lambda_1^2 + \lambda_2^2}{(\lambda_3 - \lambda_4)^2}.$$

Other connection forms are found in the same way, yet much easier:

$$\begin{aligned} \omega_{13} &= \Delta_3 \omega_1 - \Delta_1 \omega_3, \quad \omega_{24} = \Delta_4 \omega_2 - \Delta_2 \omega_4, \\ \omega_{14} &= \Delta_4 \omega_1 - \Delta_1 \omega_4, \quad \omega_{23} = \Delta_3 \omega_2 - \Delta_2 \omega_3. \end{aligned}$$

Finally, by the formula  $d\omega_{ij} - \sum_l \omega_{il} \wedge \omega_{jl} = -\frac{1}{2} R_{ijkl} \omega_k \wedge \omega_l$ , the sectional curvatures

are computed out:

$$\begin{aligned}\frac{1}{2}R_{1313} &= -E_1(\Delta_1) - E_3(\Delta_3) + \Delta_1^2 + \Delta_3^2 + I_{12}\Delta_2^2 + I_{34}\Delta_4^2, \\ \frac{1}{2}R_{2424} &= -E_2(\Delta_2) - E_4(\Delta_4) + \Delta_2^2 + \Delta_4^2 + I_{12}\Delta_1^2 + I_{34}\Delta_3^2, \\ \frac{1}{2}R_{1414} &= -E_1(\Delta_1) - E_4(\Delta_4) + \Delta_1^2 + \Delta_4^2 + I_{12}\Delta_2^2 + I_{34}\Delta_3^2, \\ \frac{1}{2}R_{2323} &= -E_2(\Delta_2) - E_3(\Delta_3) + \Delta_2^2 + \Delta_3^2 + I_{12}\Delta_1^2 + I_{34}\Delta_4^2.\end{aligned}$$

Here  $E_i(\Delta_j)$  is understood as the action of tangent vector  $E_i$  on the function  $\Delta_j$ , and  $\Delta_i^2$  is the square of  $\Delta_i$ . As a corollary,

$$R_{1313} + R_{2424} - R_{1414} - R_{2323} = 0.$$

But on the other hand, (4) implies  $R_{ijij} = \lambda_i\lambda_j + A_{ii} + A_{jj}$  when  $i \neq j$ . Substitute this into the final result above, we find

$$R_{1313} + R_{2424} - R_{1414} - R_{2323} = (\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4) = 0.$$

This contradicts our assumption  $\lambda_1 \neq \pm\lambda_2, \lambda_3 \neq \pm\lambda_4$ . Thus we have proved that the possibilities other than  $B = \pm\bar{B}$  could not happen. This completes the proof to Proposition 7.3.  $\square$

## 8 Deformable hypersurfaces with one principal curvature of multiplicity $n - 1$

In this section and the next one we make use of the following convention on the range of indices:

$$1 \leq i, j, k \leq n; \quad 3 \leq \alpha, \beta, \gamma \leq n.$$

**Proposition 8.1.** *Let  $f, \bar{f} : M^n \rightarrow R^{n+1}$  ( $n \geq 4$ ) be two hypersurfaces without umbilics. Suppose that their Möbius metrics are equal, and one principal curvature of  $B$  has multiplicity  $n - 1$  everywhere (this means that  $(M^n, g)$  is conformally flat). Then either  $f(M^n)$  is Möbius congruent to  $\bar{f}(M^n)$ , or  $f(M^n)$  has constant Möbius curvature.*

*Proof.* Since one of principal curvatures of  $B$  has multiplicity  $n - 1$  everywhere, from the algebraic Theorem 6.1, locally we can choose an orthonormal basis  $\{E_1, \dots, E_n\}$  with respect to  $g$  such that

$$\{\bar{B}_{ij}\} = \text{diag}\left(\begin{bmatrix} \bar{B}_{11} & \bar{B}_{12} \\ \bar{B}_{21} & \bar{B}_{22} \end{bmatrix}, \bar{\mu}, \dots, \bar{\mu}\right), \{B_{ij}\} = \text{diag}(\lambda, \mu, \dots, \mu), .$$

where  $\lambda \neq \mu$ . From (6) we have  $\lambda + (n-1)\mu = 0$  and  $\lambda^2 + (n-1)\mu^2 = \frac{n-1}{n}$ . So

$$\{B_{ij}\} = \text{diag} \left( \frac{n-1}{n}, \frac{-1}{n}, \dots, \frac{-1}{n} \right).$$

Since  $\bar{\mu} = \pm\mu$ , up to change of the normal direction we may assume  $\bar{\mu} = \mu = \frac{-1}{n}$ . Apply formula (6) to  $\bar{B}_{ij}$ . We know that the sub-matrices

$$\begin{pmatrix} \bar{B}_{11} & \bar{B}_{12} \\ \bar{B}_{21} & \bar{B}_{22} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{n-1}{n} & 0 \\ 0 & \frac{-1}{n} \end{pmatrix}$$

have equal traces and equal norms, and  $B, \bar{B}$  share equal eigenvalues. At any point of  $M^n$  there exists a suitable  $P = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \in SO(2)$  such that

$$(66) \quad \begin{pmatrix} \bar{B}_{11} & \bar{B}_{12} \\ \bar{B}_{21} & \bar{B}_{22} \end{pmatrix} = P^{-1} \begin{pmatrix} \frac{n-1}{n} & 0 \\ 0 & \frac{-1}{n} \end{pmatrix} P = \begin{pmatrix} \cos^2 \theta - \frac{1}{n} & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta - \frac{1}{n} \end{pmatrix}$$

We summarize some intermediate results as

**Lemma 8.2.** *For hypersurface  $f$ , the coefficients of tensor  $B, \nabla B, C$  under the orthonormal basis  $\{E_1, \dots, E_n\}$  satisfy*

$$\begin{aligned} B_{1j,j} &= -C_{1,j}, j > 1; \quad \text{otherwise, } B_{ij,k} = 0. \\ C_k &= 0, k > 1; \quad \omega_{1j} = -C_{1,j}, j > 1. \\ R_{1i1i} - C_{1,1} + C_1^2 &= 0, j > 1; \quad R_{1iji} - C_{1,j} = 0, j > 1. \end{aligned}$$

where  $\{\omega_1, \dots, \omega_n\}$  are the dual basis, and  $\{\omega_{ij}\}$  are its connection forms.

*Proof.* From  $dB_{ij} + \sum_k B_{kj}\omega_{ki} + B_{ik}\omega_{kj} = \sum_k B_{ij,k}\omega_k$  and (3) we get the first four equalities. From  $d\omega_{1i} - \sum_k \omega_{1k} \wedge \omega_{ki} = -\frac{1}{2} \sum_{kl} R_{1ikl}\omega_k \wedge \omega_l$  and invoking the proved equalities we get the equalities on the curvature tensor.  $\square$

In order to prove Proposition (8.1) We have to consider the following two cases:

**Case I,**  $\bar{B}_{12} \equiv 0$ ;

**Case II,**  $\bar{B}_{12} \neq 0$ .

First we consider Case I. Since  $\bar{B}_{12} = -\cos \theta \sin \theta \equiv 0$ , so  $\sin \theta = 0$  or  $\cos \theta = 0$ . If  $\sin \theta = 0$ , then  $\bar{B} = B$ , thus  $f$  is Möbius congruent to  $\bar{f}$ . Next we assume that  $\cos \theta = 0$ . From (9) we get

$$(\bar{B}_{ij}) = \text{diag}(\mu, \lambda, \mu, \dots, \mu) = \text{diag} \left( \frac{-1}{n}, \frac{n-1}{n}, \frac{-1}{n}, \dots, \frac{-1}{n} \right).$$

For hypersurface  $\bar{f} : M^n \rightarrow S^{n+1}$ , since  $g = \bar{g}$ , we may use the same dual basis  $\{\omega_i\}$  with the same connection forms. Similar to Lemma 8.2 we get

$$\begin{aligned}\bar{B}_{2i,i} &= -\bar{C}_2; \quad \text{otherwise } \bar{B}_{ij,k} = 0; \\ \omega_{1j} &= -\bar{C}_2\omega_j, j \neq 2; \quad \bar{C}_i = 0, i \neq 2.\end{aligned}$$

Compared with Lemma 8.2 we see  $-C_1\omega_2 = \omega_{12} = -\omega_{21} = \bar{C}_2\omega_1$ . Therefore,  $C_1 = \bar{C}_2 = 0$ , and the Möbius forms of both  $f$  and  $\bar{f}$  vanish:

$$C = 0, \bar{C} = 0.$$

Thus both  $f$  and  $\bar{f}$  are Möbius isoparametric hypersurfaces with two distinct principal curvatures. Since  $\omega_{1i} = \omega_{2i} = 0$  and  $R_{1212} = 0$ , from [19],  $f$  and  $\bar{f}$  are Möbius equivalent to the circular cylinder  $S^1(1) \times R^{n-1} \subset R^{n+1}$ , hence congruent to each other. This completes the proof to Proposition 8.1 for Case I. (In particular this is not a Möbius deformable case.)

Next we consider Case II,  $\bar{B}_{12} = -\sin \theta \cos \theta \neq 0$ .

Since  $B_{ij} = \frac{-1}{n}\delta_{ij}$ ,  $2 \leq i, j \leq n$ ,  $\bar{B}_{\alpha\beta} = \frac{-1}{n}\delta_{\alpha\beta}$ , We can rechoose  $\{E_2, \dots, E_n\}$  such that

$$(A_{ij}) = \begin{pmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1n} \\ A_{21} & a_2 & 0 & \cdots & 0 \\ A_{31} & 0 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{n1} & 0 & 0 & \cdots & a_n \end{pmatrix}, (\bar{A}_{ij}) = \begin{pmatrix} \bar{A}_{11} & \bar{A}_{12} & \bar{A}_{13} & \bar{A}_{14} & \cdots & \bar{A}_{1n} \\ \bar{A}_{21} & \bar{A}_{22} & \bar{A}_{23} & \bar{A}_{24} & \cdots & \bar{A}_{2n} \\ \bar{A}_{31} & \bar{A}_{32} & \bar{a}_3 & 0 & \cdots & 0 \\ \bar{A}_{41} & \bar{A}_{42} & 0 & \bar{a}_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{A}_{n1} & \bar{A}_{n2} & 0 & 0 & \cdots & \bar{a}_n \end{pmatrix}.$$

Noting that Lemma 8.2 holds under this basis, using  $R_{1i1i} - C_{1,1} + C_1^2 = 0$  and (4) we get that

$$a_2 = a_3 = \cdots = a_n, \bar{a}_3 = \cdots = \bar{a}_n.$$

In formula (4), Let  $i = 2, k = \alpha, j = l = 1$  and  $i = k = \alpha, k = l = \beta$  we get that

$$\bar{A}_{\alpha 2} = \bar{A}_{2\alpha} = 0, a_2 = a_3 = \cdots = a_n = \bar{a}_3 = \cdots = \bar{a}_n.$$

Thus we have

$$(67) \quad (A_{ij}) = \begin{pmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1n} \\ A_{21} & a_2 & 0 & \cdots & 0 \\ A_{31} & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{n1} & 0 & 0 & \cdots & a_2 \end{pmatrix} (\bar{A})_{ij} = \begin{pmatrix} \bar{A}_{11} & \bar{A}_{12} & \bar{A}_{13} & \cdots & \bar{A}_{1n} \\ \bar{A}_{21} & \bar{A}_{22} & 0 & \cdots & 0 \\ \bar{A}_{31} & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{A}_{n1} & 0 & 0 & \cdots & a_2 \end{pmatrix}.$$

Since  $\bar{B}_{1\alpha} = 0, \bar{B}_{2\alpha} = 0, \bar{B}_{\alpha\beta} = \frac{-1}{n}\delta_{\alpha\beta}$ , we do covariant differentiation to find

$$(68) \quad \begin{aligned} (\bar{B}_{11} + \frac{1}{n})\omega_{1\alpha} + \bar{B}_{12}\omega_{2\alpha} &= \sum_k \bar{B}_{1\alpha,k}\omega_k; \\ \bar{B}_{12}\omega_{1\alpha} + (\bar{B}_{22} + \frac{1}{n})\omega_{2\alpha} &= \sum_k \bar{B}_{2\alpha,k}\omega_k; \\ \bar{B}_{\alpha\beta,k} &= 0, \quad \forall \alpha, \beta, k. \end{aligned}$$

Using  $E_\beta(\bar{B}_{\alpha\alpha}) = \bar{B}_{\alpha\alpha,\beta}$ , (3) and (68), we get that

$$(69) \quad \bar{C}_\alpha = 0, \bar{B}_{1\alpha,\alpha} = -\bar{C}_1, \bar{B}_{2\alpha,\alpha} = -\bar{C}_2, \bar{B}_{1\alpha,\beta} = \bar{B}_{2\alpha,\beta} = 0, \alpha \neq \beta.$$

Thus from (9) and (68) we have

$$(70) \quad \begin{aligned} \cos^2 \theta \omega_{1\alpha} + \sin \theta \cos \theta \omega_{2\alpha} &= \bar{B}_{1\alpha,1}\omega_1 + \bar{B}_{1\alpha,2}\omega_2 + \bar{B}_{1\alpha,\alpha}\omega_\alpha; \\ \sin \theta \cos \theta \omega_{1\alpha} + \sin^2 \theta \omega_{2\alpha} &= \bar{B}_{2\alpha,1}\omega_1 + \bar{B}_{2\alpha,2}\omega_2 + \bar{B}_{2\alpha,\alpha}\omega_\alpha. \end{aligned}$$

Using  $E_\beta(\bar{B}_{11}) = \bar{B}_{11,\alpha}, \bar{B}_{11} + \bar{B}_{22} - \frac{n-2}{n} = 0$  and (70) we derive

$$(71) \quad \begin{aligned} \bar{B}_{1\alpha,1} = \bar{B}_{2\alpha,2} = \bar{B}_{1\alpha,2} &= 0, \sin \theta \bar{C}_1 = \cos \theta \bar{C}_2, \\ \omega_{2\alpha} &= \frac{\cos^2 \theta \bar{C}_1 - \bar{C}_1}{\cos \theta \sin \theta} \omega_\alpha. \end{aligned}$$

Using  $d\bar{B}_{2\alpha,2} + \sum_k \bar{B}_{k\alpha,2}\omega_{k2} + \sum_k \bar{B}_{2k,2}\omega_{k\alpha} + \sum_k \bar{B}_{2\alpha,k}\omega_{k2} = \sum_k \bar{B}_{2\alpha,2k}\omega_k$ , (69) and (71), we get

$$\bar{B}_{2\alpha,21} = 0.$$

Similarly we can get  $\bar{B}_{2\alpha,12} = 0$ . Using Ricci identity  $\bar{B}_{2\alpha,21} - \bar{B}_{2\alpha,12} = \sum_k \bar{B}_{k\alpha} R_{k221} + \sum_k \bar{B}_{2k} R_{k\alpha 21}$  and  $R_{1\alpha 12} = A_{2\alpha} = \bar{A}_{2\alpha} = 0$  we get  $R_{2\alpha 21} = 0$ . By (4) this implies

$$\begin{aligned} A_{1\alpha} = \bar{A}_{1\alpha} &= 0, \\ \{A_{ij}\} &= \text{diag}\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & a_2 \end{bmatrix}, a_2, \dots, a_2\right), \quad \{\bar{A}_{ij}\} = \text{diag}\left(\begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}, a_2, \dots, a_2\right). \end{aligned}$$

From (6) we have

$$\bar{B}_{11} + \bar{B}_{22} = -\frac{n-2}{n}, \bar{B}_{11}^2 + \bar{B}_{22}^2 + 2\bar{B}_{12}^2 = \frac{n^2 - 2n + 2}{n^2}.$$

Using the above identity and  $E_k(\bar{B}_{ij}) + \sum_l \bar{B}_{lj}\omega_{li}(E_k) + \sum_l \bar{B}_{il}\omega_{lj}(E_k) = \bar{B}_{ij,k}$ , we get

$$\begin{aligned} \bar{B}_{11,1} + \bar{B}_{22,1} &= 0, \bar{B}_{11,2} + \bar{B}_{22,2} = 0, \\ \bar{B}_{11}\bar{B}_{11,1} + \bar{B}_{22}\bar{B}_{22,1} + 2\bar{B}_{12}\bar{B}_{12,1} &= 0, \\ \bar{B}_{11}\bar{B}_{11,2} + \bar{B}_{22}\bar{B}_{22,2} + 2\bar{B}_{12}\bar{B}_{12,2} &= 0. \end{aligned}$$

From the above equation, (6),  $\bar{B}_{11,2} = \bar{B}_{12,1} + \bar{C}_2$ , and  $\bar{B}_{22,1} = \bar{B}_{12,2} + \bar{C}_1$ , we derive

$$(72) \quad \begin{aligned} \bar{B}_{11,2} &= 2 \cos \theta \sin \theta \bar{C}_1, \bar{B}_{22,1} = 2 \cos \theta \sin \theta \bar{C}_2, \\ \bar{B}_{12,1} &= (\cos^2 \theta - \sin^2 \theta) \bar{C}_2, \bar{B}_{12,2} = (\sin^2 \theta - \cos^2 \theta) \bar{C}_1. \end{aligned}$$

Since  $E_1(\bar{B}_{12}) = E_1(\cos \theta \sin \theta) = (\cos^2 \theta - \sin^2 \theta) E_1(\theta)$  and  $E_1(\bar{B}_{12}) = \bar{B}_{12,1}$ , from (72) we get  $E_1(\theta) = \bar{C}_2$ . Similarly we have  $E_2(\theta) = C_1 - \bar{C}_1$ , thus we have

$$(73) \quad E_1(\theta) = \bar{C}_2, E_2(\theta) = C_1 - \bar{C}_1, E_\alpha(\theta) = 0.$$

Combining Lemma 5.4 and (71) we have

$$\begin{aligned} d\omega_1 &= 0, d\omega_2 = -C_1\omega_1 \wedge \omega_2, \\ d\omega_\alpha &= C_1\omega_1 \wedge \omega_\alpha + \frac{\cos^2 \theta C_1 - \bar{C}_1}{\cos \theta \sin \theta} \omega_2 \wedge \omega_\alpha + \sum_{\beta} \omega_\beta \wedge \omega_{\beta\alpha}. \end{aligned}$$

Therefore we have

$$[E_1, E_2] = C_1 E_2.$$

Using  $d\bar{C}_1 + \bar{C}_2\omega_{21} = \sum_k \bar{C}_{1,k}\omega_k$  and  $d\bar{C}_1 + \bar{C}_2\omega_{21} = \sum_k \bar{C}_{1,k}\omega_k$ , we have

$$(74) \quad \begin{aligned} E_1(\bar{C}_1) &= \bar{C}_{1,1}, E_2(\bar{C}_1) = \bar{C}_{1,2} - C_1\bar{C}_2, \\ E_1(\bar{C}_2) &= \bar{C}_{2,1}, E_2(\bar{C}_2) = \bar{C}_{2,2} + C_1\bar{C}_1. \end{aligned}$$

Using  $[E_1, E_2](\theta) = C_1 E_2(\theta)$ , (73) and (74) we get that

$$(75) \quad \bar{C}_{1,1} + \bar{C}_{2,2} = C_{1,1} - C_1^2 = R_{1\alpha 1\alpha}.$$

Combining  $\sin \theta \bar{C}_1 = \cos \theta \bar{C}_2$ , (73) and (74), we obtain that

$$\bar{C}_{1,2} = \frac{\cos \theta}{\sin \theta} (\bar{C}_1^2 + \bar{C}_2^2 + \bar{C}_{2,2}), \bar{C}_{2,1} = \frac{\sin \theta}{\cos \theta} (\bar{C}_1^2 + \bar{C}_2^2 + \bar{C}_{1,1})$$

From above formula and (3), we have

$$(76) \quad R_{1\alpha 2\alpha} = \frac{\cos^2 \theta - \sin^2 \theta}{\sin \theta \cos \theta} (\bar{C}_1^2 + \bar{C}_2^2) + \frac{\cos \theta}{\sin \theta} \bar{C}_{2,2} - \frac{\sin \theta}{\cos \theta} \bar{C}_{1,1}.$$

Compute the covariant differentiation of  $\bar{B}_{1\alpha,\alpha}, \bar{B}_{1\alpha,1}$ . By (69) and the Ricci identity,

$$(77) \quad R_{1\alpha 2\alpha} + \frac{\sin \theta}{\cos \theta} R_{2\alpha 2\alpha} + \frac{\sin^2 \theta - \cos^2 \theta}{\sin^2 \theta \cos^2 \theta} \bar{C}_1 \bar{C}_2 = \frac{\bar{C}_{2,2}}{\cos \theta \sin \theta}.$$

Similarly using Ricci identity  $\bar{B}_{2\alpha,2\alpha} - \bar{B}_{2\alpha,\alpha 2} = \sum_k \bar{B}_{k\alpha} R_{k22\alpha} + \sum_k \bar{B}_{2k} R_{k\alpha 2\alpha}$  we get

$$(78) \quad R_{1\alpha 2\alpha} + \frac{\cos \theta}{\sin \theta} R_{1\alpha 1\alpha} + \frac{\cos^2 \theta - \sin^2 \theta}{\sin^2 \theta \cos^2 \theta} \bar{C}_1 \bar{C}_2 = \frac{\bar{C}_{1,1}}{\cos \theta \sin \theta}.$$

Sum (77) and (78). Using (75) we get that

$$(79) \quad 2R_{1\alpha 2\alpha} + \frac{\sin \theta}{\cos \theta}(R_{2\alpha 2\alpha} - R_{1\alpha 1\alpha}) = 0.$$

Note that  $\sin \theta \bar{C}_1 = \cos \theta \bar{C}_2$ , combining (76), (77) and (78), we obtain that

$$(80) \quad 2R_{1\alpha 2\alpha} - \frac{\cos \theta}{\sin \theta}(R_{2\alpha 2\alpha} - R_{1\alpha 1\alpha}) = 0.$$

From (79) and (80), we get that

$$(81) \quad R_{1\alpha 1\alpha} - R_{2\alpha 2\alpha} = 0, \quad R_{1\alpha 2\alpha} = 0.$$

Therefore from (4) we get  $R_{2\alpha 2\alpha} = R_{\alpha\beta\alpha\beta}$ . Hence  $R_{ijkl} = R_{2\alpha 2\alpha}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{kj})$ . By Schur's theorem  $(M^n, g)$  is of constant curvature. This completes the proof.  $\square$

## 9 Deformable hypersurfaces with one principal curvature of multiplicity $n - 2$

This section is devoted to the proof of the following

**Proposition 9.1.** *Let  $f, \bar{f} : M^n \rightarrow R^{n+1}$  ( $n \geq 4$ ) be two hypersurfaces without umbilics. Suppose their Möbius metrics are equal, and one of principal curvatures of  $B$  has multiplicities  $n - 2$  everywhere. Assume that  $f(M^n)$  is NOT Möbius congruent to  $\bar{f}(M^n)$ . Then it must be either of the following three cases:*

(1)  $f(M^n)$  is congruent to part of  $L^2 \times R^{n-2}$  and  $\bar{f}(M^n)$  is congruent to part of  $\bar{L}^2 \times R^{n-2}$ , where  $L^2$  and  $\bar{L}^2$  are a pair of isometric Bonnet surface in  $R^3$ .

(2)  $f(M^n)$  is congruent to part of  $CL^2 \times R^{n-3}$  where  $CL^2 \subset R^4$  is a cone over  $L^2 \subset S^3$ , and  $\bar{f}(M^n)$  is congruent to part of  $C\bar{L}^2 \times R^{n-3}$ .  $L^2$  and  $\bar{L}^2$  form a Bonnet pair in  $S^3$ .

(3)  $f(M^n)$  is a rotation hypersurfaces over  $L^2 \subset R_+^3$ , and  $\bar{f}(M^n)$  is a rotation hypersurfaces over  $\bar{L}^2 \subset R_+^3$ , where  $L^2$  and  $\bar{L}^2$  form a Bonnet pair in hyperbolic half-space  $R_+^3$ .

Recall that we have adopted the following convention on the range of indices as the last section:

$$1 \leq i, j, k \leq n; \quad 3 \leq \alpha, \beta, \gamma \leq n.$$

Since one of principal curvatures of  $B$  has multiplicity  $(n - 2)$  everywhere, by Theorem 6.1 we can assume without loss of generality that there exists a local orthonormal basis  $\{E_1, \dots, E_n\}$  for  $(M^n, g)$  which is shared by  $f, \bar{f}$ , such that

$$(82) \quad \{B_{ij}\} = \text{diag}(\lambda_1, \lambda_2, \mu, \dots, \mu); \quad \{\bar{B}_{ij}\} = \text{diag}\left(\begin{bmatrix} \bar{B}_{11} & \bar{B}_{12} \\ \bar{B}_{21} & \bar{B}_{22} \end{bmatrix}, \mu, \dots, \mu\right)$$



where  $\lambda_1 \neq \mu, \lambda_2 \neq \mu$ . By the identities (6) we know that the sub-matrices

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \bar{B}_{11} & \bar{B}_{12} \\ \bar{B}_{21} & \bar{B}_{22} \end{pmatrix}$$

have equal traces and equal norms. Hence  $B, \bar{B}$  share the same eigenvalues. Therefore, the highest multiplicity of principal curvatures of  $\bar{f}$  is also  $n - 2$ . Denote  $\bar{B}_{11} = \bar{\lambda}_1, \bar{B}_{22} = \bar{\lambda}_2$ .

We assert that  $\lambda_1 \neq \lambda_2$  and  $\lambda_1 \neq \bar{\lambda}_1$  on an open dense subset of  $M$ . Otherwise, suppose  $\lambda_1 = \lambda_2$  on an open subset. Then the 2 by 2 sub-matrices share the same eigenvalues  $\lambda_1 = \lambda_2$ , hence both be scalar matrix  $\text{diag}(\lambda_1, \lambda_1)$ . We get  $B = \bar{B}$  on an open subset. So these two hypersurfaces are Möbius equivalent. Contradiction. If  $\lambda_1 = \bar{\lambda}_1$  on an open subset we will get a similar contradiction.

From now on, without loss of generality we assume that on  $M^n$

$$(83) \quad \lambda_1 \neq \mu, \lambda_2 \neq \mu, \lambda_1 \neq \lambda_2, \lambda_1 \neq \bar{\lambda}_1.$$

By (6) they satisfy

$$(84) \quad \begin{aligned} \lambda_1 + \lambda_2 + (n - 2)\mu &= \bar{\lambda}_1 + \bar{\lambda}_2 + (n - 2)\mu = 0, \\ \lambda_1^2 + \lambda_2^2 + (n - 2)\mu^2 &= \bar{\lambda}_1^2 + \bar{\lambda}_2^2 + 2\bar{B}_{12}^2 + (n - 2)\mu^2 = \frac{n - 1}{n}. \end{aligned}$$

Choose the dual basis  $\{\omega_i\}$  with connection forms  $\omega_{ij}$  satisfying  $d\omega_i = \sum_k \omega_{ik} \wedge \omega_k$ ,  $\omega_{ij} = -\omega_{ji}$ . It follows from  $dB_{ij} + \sum_k B_{kj}\omega_{ki} + \sum_k B_{ik}\omega_{kj} = \sum_k B_{ij,k}\omega_k$  that

$$(85) \quad \begin{aligned} 0 &= B_{\alpha\beta,i} = B_{1\alpha,\beta} = B_{2\alpha,\beta}, \quad \alpha \neq \beta; \\ (\lambda_1 - \mu)\omega_{1\alpha} &= B_{1\alpha,1}\omega_1 + B_{1\alpha,2}\omega_2 + B_{1\alpha,\alpha}\omega_\alpha; \\ (\lambda_2 - \mu)\omega_{2\alpha} &= B_{2\alpha,1}\omega_1 + B_{2\alpha,2}\omega_2 + B_{2\alpha,\alpha}\omega_\alpha; \\ (\lambda_1 - \lambda_2)\omega_{12} &= \sum_k B_{12,k}\omega_k. \end{aligned}$$

Using  $d\bar{B}_{ij} + \sum_k \bar{B}_{kj}\omega_{ki} + \sum_k \bar{B}_{ik}\omega_{kj} = \sum_k \bar{B}_{ij,k}\omega_k$  in a similar way we obtain

$$(86) \quad \begin{aligned} 0 &= \bar{B}_{\alpha\beta,i} = \bar{B}_{1\alpha,\beta} = \bar{B}_{2\alpha,\beta}, \quad \alpha \neq \beta; \\ (\bar{\lambda}_1 - \mu)\omega_{1\alpha} + \bar{B}_{12}\omega_{2\alpha} &= \bar{B}_{1\alpha,1}\omega_1 + \bar{B}_{1\alpha,2}\omega_2 + \bar{B}_{1\alpha,\alpha}\omega_\alpha; \\ (\bar{\lambda}_2 - \mu)\omega_{2\alpha} + \bar{B}_{12}\omega_{1\alpha} &= \bar{B}_{2\alpha,1}\omega_1 + \bar{B}_{2\alpha,2}\omega_2 + \bar{B}_{2\alpha,\alpha}\omega_\alpha; \\ d\bar{B}_{12} + (\bar{\lambda}_1 - \bar{\lambda}_2)\omega_{12} &= \sum_k \bar{B}_{12,k}\omega_k. \end{aligned}$$

Comparing (85) with (86) yields

$$\begin{aligned}
(87) \quad & \bar{B}_{1\alpha,1} = \frac{\bar{\lambda}_1 - \mu}{\lambda_1 - \mu} B_{1\alpha,1} + \frac{\bar{B}_{12}}{\lambda_2 - \mu} B_{2\alpha,1}; \quad \bar{B}_{2\alpha,1} = \frac{\bar{\lambda}_2 - \mu}{\lambda_2 - \mu} B_{2\alpha,1} + \frac{\bar{B}_{12}}{\lambda_1 - \mu} B_{1\alpha,1}; \\
& \bar{B}_{1\alpha,2} = \frac{\bar{\lambda}_1 - \mu}{\lambda_1 - \mu} B_{1\alpha,2} + \frac{\bar{B}_{12}}{\lambda_2 - \mu} B_{2\alpha,2}; \quad \bar{B}_{2\alpha,2} = \frac{\bar{\lambda}_2 - \mu}{\lambda_2 - \mu} B_{2\alpha,2} + \frac{\bar{B}_{12}}{\lambda_1 - \mu} B_{1\alpha,2}; \\
& \bar{B}_{1\alpha,\alpha} = \frac{\bar{\lambda}_1 - \mu}{\lambda_1 - \mu} B_{1\alpha,\alpha} + \frac{\bar{B}_{12}}{\lambda_2 - \mu} B_{2\alpha,\alpha}; \quad \bar{B}_{2\alpha,\alpha} = \frac{\bar{\lambda}_2 - \mu}{\lambda_2 - \mu} B_{2\alpha,\alpha} + \frac{\bar{B}_{12}}{\lambda_1 - \mu} B_{1\alpha,\alpha}.
\end{aligned}$$

Another corollary of (86),(87) and (3) is

$$(88) \quad B_{\alpha\alpha,\beta} = C_\beta = \bar{C}_\beta = \bar{B}_{\alpha\alpha,\beta} = B_{\beta\beta,\beta} = \bar{B}_{\beta\beta,\beta}, \quad \forall \alpha \neq \beta.$$

Taking the covariant derivatives for the identities (84) and invoking (88), we have

$$\begin{aligned}
(89) \quad & B_{11,\alpha} + B_{22,\alpha} = (2-n)C_\alpha, \quad \lambda_1 B_{11,\alpha} + \lambda_2 B_{22,\alpha} = (2-n)\mu C_\alpha; \\
& \bar{B}_{11,\alpha} + \bar{B}_{22,\alpha} = (2-n)\bar{C}_\alpha, \quad \bar{\lambda}_1 \bar{B}_{11,\alpha} + \bar{\lambda}_2 \bar{B}_{22,\alpha} = (2-n)\mu \bar{C}_\alpha - 2\bar{B}_{12}\bar{B}_{12,\alpha}.
\end{aligned}$$

The two equations in the first line have solution

$$(90) \quad B_{11,\alpha} = \frac{\mu - \lambda_1}{\lambda_2 - \lambda_1} (2-n)C_\alpha, \quad B_{22,\alpha} = \frac{\lambda_2 - \mu}{\lambda_2 - \lambda_1} (2-n)C_\alpha.$$

Take the sum of the first and the fourth equations in (87) and insert (90) into it. By (3), the first equation in (89), and the identities (84), the result is as below after simplification:

$$(91) \quad (\lambda_1 - \lambda_2)\mu \bar{B}_{12} B_{12,\alpha} = \frac{n-1}{n} (\bar{\lambda}_1 - \lambda_1) C_\alpha.$$

On the other hand, take the difference between the second and the third equations in (87). After simplification as before we get

$$(92) \quad (\lambda_1 - \bar{\lambda}_1)(\lambda_1 - \lambda_2)\mu B_{12,\alpha} = \frac{n-1}{n} \bar{B}_{12} C_\alpha.$$

It follows from (91)(92) that

$$(93) \quad C_\alpha = \bar{C}_\alpha = 0, \quad \forall \alpha.$$

Otherwise there will be  $B_{12,\alpha}^2 = -(\bar{\lambda}_1 - \lambda_1)^2$  which is impossible. As a corollary of (88)(90) and (93),

$$(94) \quad B_{11,\alpha} = B_{22,\alpha} = B_{\beta\beta,\alpha} = \bar{B}_{\beta\beta,\alpha} = 0, \quad B_{1\alpha,1} = B_{2\alpha,2} = 0, \quad \forall \alpha, \beta.$$

We emphasize that (92)(83) and  $C_\alpha = 0$  implies  $\mu \cdot B_{12,\alpha} = 0$ . Now we divide the proof into two cases.

**Case I,**  $B_{12,\alpha} = 0$ , for all  $\alpha$ .

**Case II,**  $B_{12,\alpha} \neq 0$ , for some  $\alpha$ .

First we consider Case I. Since  $C_\alpha = 0, B_{12,\alpha} = 0$ , from the Reduction Theorem 5.1 we know that  $f$  is Möbius equivalent to a hypersurface given by Example (3.1),(3.3) or (3.5) when  $Q = 0, Q < 0$  or  $Q > 0$ , respectively.

We define  $\omega_{ij} = \sum_k \Gamma_{jk}^i \omega_k$ . From (85) and the definition of  $Q$  in the proof of Theorem 5.1, we get that

$$Q = 2A_{\alpha\alpha} + \mu^2 + (\Gamma_{\alpha\alpha}^1)^2 + (\Gamma_{\alpha\alpha}^2)^2.$$

On the other hand,  $R_{\alpha\beta\alpha\beta} = \mu^2 + 2A_{\alpha\alpha} = \mu^2 + 2\bar{A}_{\alpha\alpha}$ , so  $A_{\alpha\alpha} = \bar{A}_{\alpha\alpha}$ . Therefore

$$Q = \bar{Q},$$

and  $f, \bar{f}$  are congruent to two cylinders, or two cones, or two rotational hypersurfaces over some surfaces in a 3-dimensional space form. According to Remark 3.7 and (12), in either case they share the same metric

$$g = \left[ 4H_u^2 - \frac{2n}{n-1}(K_u + c) \right] (I_u + I_{N^{n-2}(c)}).$$

So they must share the same surface metric  $I_u$ , hence the same surface curvature  $K_u$ , hence also the same mean curvature. Therefore they come from a Bonnet pair in the corresponding 3-space. This finishes our proof of Proposition 9.1 in Case I.

Next we consider Case II where  $B_{12,\alpha} \neq 0$  for some  $\alpha$ . We have the following results.

**Proposition 9.2.** *Let  $f, \bar{f} : M^n \rightarrow R^{n+1}$  ( $n \geq 4$ ) be two hypersurfaces without umbilics. Suppose their Möbius metrics are equal, and one of principal curvatures of  $B$  has multiplicities  $n-2$  everywhere. We can assume that there exists a local orthonormal basis  $\{E_1, \dots, E_n\}$  for  $(M^n, g)$  which is shared by  $f, \bar{f}$ , such that*

$$\{B_{ij}\} = \text{diag}(\lambda_1, \lambda_2, \mu, \dots, \mu); \{\bar{B}_{ij}\} = \text{diag}\left(\begin{bmatrix} \bar{B}_{11} & \bar{B}_{12} \\ \bar{B}_{21} & \bar{B}_{22} \end{bmatrix}, \mu, \dots, \mu\right)$$

where  $\lambda_1 \neq \mu, \lambda_2 \neq \mu$ . If  $B_{12,\alpha} \neq 0$ , for some  $\alpha$ . Then there exist an diffeomorphism  $\psi : M^n \rightarrow M^n$  and a Möbius transformation  $\Phi$  such that  $\Phi \circ f = \bar{f} \circ \psi : M^n \rightarrow R^{n+1}$ . Moreover,  $f$  is Möbius equivalent to the minimal hypersurface defined by

$$x = (x_1, x_2) : M^n = N^3 \times H^{n-3}\left(-\frac{n-1}{6n}\right) \rightarrow S^{n+1},$$

where

$$x_1 = \frac{y_1}{y_0}, x_2 = \frac{y_2}{y_0}, y_0 \in R^+, y_1 \in R^5, y_2 \in R^{n-3}.$$

Here  $y_1 : N^3 \rightarrow S^4(\sqrt{\frac{6n}{n-1}}) \hookrightarrow R^5$  is Cartan's minimal isoparametric hypersurface in  $S^4(\sqrt{\frac{6n}{n-1}})$  with three principal curvatures, and  $(y_0, y_2) : H^{n-3}(-\frac{n-1}{6n}) \hookrightarrow R_1^{n-2}$  is the standard embedding of the hyperbolic space of sectional curvature  $-\frac{n-1}{6n}$  into the  $(n-2)$ -dimensional Lorentz space with  $-y_0^2 + y_2^2 = \frac{6n}{n-1}$ .

*Proof.* Since  $B_{\alpha\beta} = \bar{B}_{\alpha\beta} = 0$ , We can assume that

$$(95) \quad B_{12,3} \neq 0, \quad B_{12,\alpha} = 0, \alpha \neq 3.$$

From (89) we have

$$(96) \quad \begin{aligned} \omega_{12} &= \frac{-C_2}{2\lambda_1}\omega_1 - \frac{C_1}{2\lambda_1}\omega_2 + \frac{B_{12,3}}{2\lambda_1}\omega_3; \\ \omega_{13} &= \frac{B_{12,3}}{\lambda_1}\omega_2 - \frac{C_1}{\lambda_1}\omega_3, \quad \omega_{23} = \frac{B_{12,3}}{\lambda_2}\omega_2 - \frac{C_2}{\lambda_2}\omega_3; \\ \omega_{1\alpha} &= \frac{-C_1}{\lambda_1}\omega_\alpha, \quad \omega_{2\alpha} = \frac{-C_2}{\lambda_2}\omega_\alpha, \alpha > 3. \end{aligned}$$

Since  $C_\alpha = 0$ , using  $dC_i + \sum_m C_m \omega_{mi} = \sum_m C_{i,m} \omega_m$  and (96), we get

$$(97) \quad \begin{aligned} C_{\alpha,\alpha} &= \frac{C_2^2 - C_1^2}{\lambda_1}, \quad C_{\alpha,k} = 0, k \neq \alpha, \alpha > 3; \\ C_{3,3} &= \frac{C_2^2 - C_1^2}{\lambda_1}, C_{3,1} = \frac{B_{12,3}C_2}{\lambda_2}, C_{3,2} = \frac{B_{12,3}C_1}{\lambda_1}, C_{3,\alpha} = 0, \alpha > 3. \end{aligned}$$

Differentiating the equations (96), we get

$$(98) \quad \begin{aligned} \frac{-1}{2} \sum_{kl} R_{12kl} \omega_k \wedge \omega_l &= \frac{-1}{2\lambda_1} \sum_m [C_{2,m} \omega_m \wedge \omega_1 + C_{1,m} \omega_m \wedge \omega_2] - 3 \frac{B_{12,3}C_1}{2\lambda_1^2} \omega_1 \wedge \omega_3 \\ &+ [\frac{C_1^2 + C_2^2}{2\lambda_1^2} + 2 \frac{B_{12,3}^2}{\lambda_1^2}] \omega_1 \wedge \omega_2 + 3 \frac{B_{12,3}C_2}{2\lambda_1^2} \omega_2 \wedge \omega_3 + \frac{dB_{12,3}}{2\lambda_1} \wedge \omega_3. \end{aligned}$$

$$(99) \quad \begin{aligned} \frac{-1}{2} \sum_{kl} R_{13kl} \omega_k \wedge \omega_l &= \frac{1}{\lambda_1} dB_{12,3} \wedge \omega_2 - \frac{1}{\lambda_1} \sum_m C_{1,m} \omega_m \wedge \omega_3 - 2 \frac{B_{12,3}C_1}{\lambda_1^2} \omega_1 \wedge \omega_2 \\ &+ [\frac{C_1^2 + C_2^2}{2\lambda_1^2} - \frac{B_{12,3}^2}{\lambda_1^2}] \omega_1 \wedge \omega_3. \end{aligned}$$

$$(100) \quad \begin{aligned} \frac{-1}{2} \sum_{kl} R_{23kl} \omega_k \wedge \omega_l &= \frac{1}{\lambda_1} dB_{12,3} \wedge \omega_1 - \frac{1}{\lambda_2} \sum_m C_{2,m} \omega_m \wedge \omega_3 + 2 \frac{B_{12,3}C_2}{\lambda_1^2} \omega_1 \wedge \omega_2 \\ &+ [\frac{C_1^2 + C_2^2}{2\lambda_1^2} - \frac{B_{12,3}^2}{\lambda_1^2}] \omega_2 \wedge \omega_3. \end{aligned}$$

$$(101) \quad \begin{aligned} \frac{-1}{2} \sum_{kl} R_{1\alpha kl} \omega_k \wedge \omega_l &= \frac{-1}{\lambda_1} \sum_m C_{1,m} \omega_m \wedge \omega_\alpha + \frac{C_1^2 + C_2^2}{\lambda_1^2} \omega_1 \wedge \omega_\alpha \\ &\quad - \frac{B_{12,3} C_2}{\lambda_1^2} \omega_3 \wedge \omega_\alpha - \frac{B_{12,3}}{\lambda_1} \omega_2 \wedge \omega_{3\alpha}. \end{aligned}$$

$$(102) \quad \begin{aligned} \frac{-1}{2} \sum_{kl} R_{2\alpha kl} \omega_k \wedge \omega_l &= \frac{-1}{\lambda_1} \sum_m C_{2,m} \omega_m \wedge \omega_\alpha + \frac{C_1^2 + C_2^2}{\lambda_1^2} \omega_2 \wedge \omega_\alpha \\ &\quad - \frac{B_{12,3} C_1}{\lambda_1^2} \omega_3 \wedge \omega_\alpha + \frac{B_{12,3}}{\lambda_1} \omega_1 \wedge \omega_{3\alpha}. \end{aligned}$$

Comparing the coefficients of  $\omega_3 \wedge \omega_\alpha$  on both sides of (99), and using (4) we obtain

$$(103) \quad A_{1\alpha} = 0, \quad A_{3\alpha} = 0, \quad E_\alpha(B_{12,3}) = 0, \quad \alpha > 3.$$

Similarly from (96), (100), (101) and (102), we have

$$(104) \quad \begin{aligned} A_{2\alpha} &= 0, \quad A_{\alpha\beta} = 0, \quad \alpha, \beta > 3, \quad \alpha \neq \beta; \\ \omega_{3\alpha}(E_1) &= 0, \quad \omega_{3\alpha}(E_2) = 0, \quad \omega_{3\alpha}(E_3) = 0, \quad \omega_{3\alpha}(E_\beta) = 0, \quad \alpha, \beta > 3, \quad \alpha \neq \beta; \\ R_{1\alpha 1\alpha} &= -\frac{C_1^2 + C_2^2}{\lambda_1^2} + \frac{C_{1,1}}{\lambda_1}, \quad R_{2\alpha 2\alpha} = -\frac{C_1^2 + C_2^2}{\lambda_1^2} + \frac{C_{2,2}}{\lambda_2}, \quad \alpha > 3; \\ R_{1313} &= \frac{B_{12,3}^2}{\lambda_1^2} - \frac{C_1^2 + C_2^2}{\lambda_1^2} + \frac{C_{1,1}}{\lambda_1}, \quad R_{2323} = \frac{B_{12,3}^2}{\lambda_1^2} - \frac{C_1^2 + C_2^2}{\lambda_1^2} + \frac{C_{2,2}}{\lambda_2}; \\ R_{1212} &= \frac{C_{1,1} - C_{2,2}}{2\lambda_1} - 2\frac{B_{12,3}^2}{\lambda_1^2} - \frac{C_1^2 + C_2^2}{2\lambda_1^2}. \\ E_2(B_{12,3}) &= \lambda_1 A_{13} - 2\frac{B_{12,3} C_2}{\lambda_1}, \quad E_1(B_{12,3}) = \lambda_2 A_{23} + 2\frac{B_{12,3} C_1}{\lambda_1}; \\ (105) \quad A_{12} &= \frac{C_{1,2}}{\lambda_1} + \frac{B_{12,3}}{\lambda_1} \omega_{3\alpha}(E_\alpha), \quad A_{12} = \frac{-C_{2,1}}{\lambda_1} - \frac{B_{12,3}}{\lambda_1} \omega_{3\alpha}(E_\alpha); \\ A_{12} &= \frac{-C_{1,2}}{\lambda_1} + \frac{-1}{\lambda_1} E_3(B_{12,3}), \quad A_{12} = \frac{C_{2,1}}{\lambda_1} + \frac{-1}{\lambda_1} E_3(B_{12,3}). \end{aligned}$$

From (105), (100) and (101), we have

$$(106) \quad E_3(B_{12,3}) = B_{12,3} \omega_{3\alpha}(E_\alpha).$$

Define  $\phi := \omega_{3\alpha}(E_\alpha) = \frac{E_3(B_{12,3})}{B_{12,3}}$ . From (99), we have

$$(107) \quad \omega_{3\alpha} = \phi \omega_\alpha.$$

Differentiating the equations (107), we get

$$(108) \quad \begin{aligned} \frac{-1}{2} \sum_{kl} R_{3\alpha kl} \omega_k \wedge \omega_l &= d\phi \wedge \omega_\alpha - \frac{C_1}{\lambda_1} \phi \omega_1 \wedge \omega_\alpha - \frac{C_2}{\lambda_2} \phi \omega_2 \wedge \omega_\alpha + \phi^2 \omega_3 \wedge \omega_\alpha \\ &\quad - \frac{B_{12,3} C_1}{\lambda_1^2} \omega_2 \wedge \omega_\alpha + \frac{C_1^2 + C_2^2}{\lambda_1^2} \omega_3 \wedge \omega_\alpha - \frac{B_{12,3} C_2}{\lambda_1^2} \omega_1 \wedge \omega_\alpha. \end{aligned}$$

Comparing the coefficients of  $\omega_1 \wedge \omega_\alpha$  and  $\omega_2 \wedge \omega_\alpha$  on both sides of (108), and using (4) we obtain

$$(109) \quad -A_{13} = E_1(\phi) - \frac{C_1}{\lambda_1}\phi - \frac{B_{12,3}C_2}{\lambda_1^2}, -A_{23} = E_2(\phi) - \frac{C_2}{\lambda_2}\phi - \frac{B_{12,3}C_2}{\lambda_1^2}.$$

Using  $dA_{ij} + \sum_m A_{mj}\omega_{mi} + \sum_m A_{im}\omega_{mj} = \sum_m A_{ij,m}\omega_m$  and (99), we obtain

$$(110) \quad \begin{aligned} A_{1\alpha,\alpha} &= (A_{\alpha\alpha} - A_{11})\frac{C_1}{\lambda_1} + A_{12}\frac{C_2}{\lambda_1} + A_{13}\omega_{3\alpha}(E_\alpha), \quad A_{1\alpha,k} = 0, k \neq \alpha; \\ A_{2\alpha,\alpha} &= (A_{22} - A_{\alpha\alpha})\frac{C_2}{\lambda_1} - A_{12}\frac{C_1}{\lambda_1} + A_{23}\omega_{3\alpha}(E_\alpha), \quad A_{2\alpha,k} = 0, k \neq \alpha; \\ A_{3\alpha,\alpha} &= (A_{33} - A_{\alpha\alpha})\omega_{3\alpha}(E_\alpha) - A_{13}\frac{C_1}{\lambda_1} + A_{23}\frac{C_2}{\lambda_1}, \quad A_{3\alpha,k} = 0, k \neq \alpha. \end{aligned}$$

On the other hands, from (100) we have

$$A_{33} - A_{\alpha\alpha} = \frac{B_{12,3}^2}{\lambda_1^2}, \alpha > 3.$$

Noting that  $E_\alpha(B_{12,3}) = 0$  and  $A_{33,\alpha} = A_{3\alpha,3} = 0$ , we get

$$(111) \quad E_\alpha(A_{\alpha\alpha}) = E_\alpha(A_{\beta\beta}) = 0, \alpha \neq \beta, \alpha, \beta > 3.$$

Combining (96), (109), (108), (110) and (111), we get

$$(112) \quad \begin{aligned} A_{1\alpha,\alpha 1} &= (A_{\alpha\alpha,1} - A_{11,1})\frac{C_1}{\lambda_1} + (A_{11} - A_{\alpha\alpha})\left[\frac{C_2^2}{\lambda_1^2} - \frac{C_{1,1}}{\lambda_1}\right] + A_{12}\frac{C_1C_2}{\lambda_1^2} \\ &\quad + A_{12,1}\frac{C_2}{\lambda_1} + A_{12}C_{2,1} + A_{13,1}\phi - A_{13}^2 + \frac{A_{13}\phi C_1}{\lambda_1} - A_{12}\frac{B_{12,3}\phi}{\lambda_1}; \\ A_{1\alpha,1\alpha} &= (2A_{\alpha\alpha,1} - A_{11,1})\frac{C_1}{\lambda_1} + A_{12,2}\frac{C_2}{\lambda_1} + A_{13,1}\phi. \end{aligned}$$

Combining (104), (112) and Ricci identity  $A_{1\alpha,1\alpha} - A_{1\alpha,\alpha 1} = \sum_m A_{m\alpha}R_{m11\alpha} + \sum_m A_{1m}R_{m\alpha 1\alpha}$ , we obtain

$$(113) \quad A_{13} = 0.$$

similarly using Ricci identity  $A_{2\alpha,2\alpha} - A_{2\alpha,\alpha 2} = \sum_m A_{m\alpha}R_{m22\alpha} + \sum_m A_{2m}R_{m\alpha 2\alpha}$ , we have

$$(114) \quad A_{23} = 0.$$

Using (113), (114) and  $dA_{ij} + \sum_m A_{mj}\omega_{mi} + \sum_m A_{im}\omega_{mj} = \sum_m A_{ij,m}\omega_m$  and (99), we obtain

$$(115) \quad A_{13,2} = (A_{11} - A_{33})\frac{B_{12,3}}{\lambda_1}, \quad A_{23,1} = (A_{22} - A_{33})\frac{B_{12,3}}{\lambda_2}.$$

From (1), we know that  $A_{13,2} = A_{23,1}$ , thus equations (115) mean that

$$(116) \quad A_{11} + A_{22} = 2A_{33}.$$

Combining (104) and (116), we obtain

$$(117) \quad 2\lambda_1^2 = 6\frac{B_{12,3}^2}{\lambda_1^2} - \frac{C_1^2 + C_2^2}{\lambda_1^2}.$$

taking derivatives for (117) along  $E_3$  and using (2) and (97), we have

$$E_3(B_{12,3}) = 0.$$

this means that  $\phi = 0$ . From (109), (113) and (114), we get

$$(118) \quad B_{12,3}\frac{C_1}{\lambda_1^2} = 0, \quad B_{12,3}\frac{C_2}{\lambda_1^2} = 0.$$

one deduced the Möbius form  $\Phi = 0$ .

Since  $\mu = 0$ , we get from (6) that  $\lambda_1 = \sqrt{\frac{n-1}{2n}}$ ,  $\lambda_2 = -\sqrt{\frac{n-1}{2n}}$ . Thus  $f$  is a Möbius isoparametric hypersurface with Möbius principal curvatures

$$\sqrt{\frac{n-1}{2n}}, -\sqrt{\frac{n-1}{2n}}, 0, \dots, 0.$$

It is then easy to show (or by the classification result in [15] of Möbius isoparametric hypersurfaces with three distinct principal curvatures) that  $f$  is Möbius equivalent to the minimal hypersurface defined by

$$x = (x_1, x_2) : M^n = N^3 \times H^{n-3}(-\frac{n-1}{6n}) \rightarrow S^{n+1},$$

where

$$x_1 = \frac{y_1}{y_0}, x_2 = \frac{y_2}{y_0}, y_0 \in R^+, y_1 \in R^5, y_2 \in R^{n-3}.$$

Here  $y_1 : N^3 \rightarrow S^4(\sqrt{\frac{6n}{n-1}}) \hookrightarrow R^5$  is Cartan's minimal isoparametric hypersurface in  $S^4(\sqrt{\frac{6n}{n-1}})$  with three principal curvatures, and  $(y_0, y_2) : H^{n-3}(-\frac{n-1}{6n}) \hookrightarrow R_1^{n-2}$  is the standard embedding of the hyperbolic space of sectional curvature  $-\frac{n-1}{6n}$  into the  $(n-2)$ -dimensional Lorentz space with  $-y_0^2 + y_2^2 = \frac{6n}{n-1}$ .

On the other hand, Since  $\bar{\mu} = \mu = 0$ , then  $B_{\alpha\alpha,1} = \bar{B}_{\alpha\alpha,1} = E_1(\bar{\mu}) = 0, B_{\alpha\alpha,2} = \bar{B}_{\alpha\alpha,2} = E_2(\bar{\mu}) = 0$ . Using (3), we have

$$\bar{B}_{1\alpha,\alpha} = -\bar{C}_1, \bar{B}_{2\alpha,\alpha} = -\bar{C}_2, B_{1\alpha,\alpha} = C_1 = 0, B_{2\alpha,\alpha} = C_2 = 0.$$

For the last two of equations (87), we get that  $\bar{\Phi} = 0$ . Since  $B, \bar{B}$  share equal eigenvalues, then  $\bar{f}$  determines the same Möbius isoparametric hypersurface as  $f$ . So  $f(M)$  is Möbius equivalent to  $\bar{f}(M)$ .

In fact, at every point of  $M^n$  there exist a suitable  $P = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \in SO(2)$  such that

$$\begin{pmatrix} \bar{B}_{11} & \bar{B}_{12} \\ \bar{B}_{21} & \bar{B}_{22} \end{pmatrix} = P^{-1} \begin{pmatrix} \sqrt{\frac{n-1}{2n}} & 0 \\ 0 & -\sqrt{\frac{n-1}{2n}} \end{pmatrix} P.$$

One can show by computation that  $\theta$  is a constant. Then one can construct a local diffeomorphism  $\psi : M^n \rightarrow M^n$  such that  $\bar{f} \circ \psi$  not only share the same metric as  $f$ , but also share the same Möbius principal curvatures and the same principal directions. Thus there exists Möbius transformation  $\Psi$  such that

$$\bar{f} \circ \psi = \Psi \circ f.$$

This is exactly the case as in Remark 1.7. Thus we do not get new Möbius deformable examples.  $\square$

Thus we have verified Proposition 9.1 in all cases and completed the proof to the Main Theorem 1.5.

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